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Abstract. The conjecture of Lehmer is proved to be true. The proof mainly relies upon: (i) the properties of the Parry Upper functions $f_{\lceil\alpha\rceil}(z)$ associated with the dynamical zeta functions $\zeta_{\lceil\alpha\rceil}(z)$ of the Rényi–Parry arithmetical dynamical systems, for α an algebraic integer α of house $\lceil\alpha\rceil$ greater than 1, (ii) the discovery of lenticuli of poles of $\zeta_{\lceil\alpha\rceil}(z)$ which uniformly equidistribute at the limit on a limit "lenticular" arc of the unit circle, when $\lceil\alpha\rceil$ tends to 1^+, giving rise to a continuous lenticular minorant $\mathrm{M}_r(\lceil\alpha\rceil)$ of the Mahler measure $\mathrm{M}(\alpha)$, (iii) the Poincaré asymptotic expansions of these poles and of this minorant $\mathrm{M}_r(\lceil\alpha\rceil)$ as a function of the dynamical degree. With the same arguments the conjecture of Schinzel-Zassenhaus is proved to be true. An inequality improving those of Dobrowolski and Voutier ones is obtained. The set of Salem numbers is shown to be bounded from below by the Perron number $\theta_{31}^{-1} = 1.08545\ldots$, dominant root of the trinomial $-1-z^{30}+z^{31}$. Whether Lehmer's number is the smallest Salem number remains open. A lower bound for the Weil height of nonzero totally real algebraic numbers, $\neq \pm 1$, is obtained (Bogomolov property). For sequences of algebraic integers of Mahler measure smaller than the smallest Pisot number, whose houses have a dynamical degree tending to infinity, the Galois orbit measures of conjugates are proved to converge towards the Haar measure on $|z| = 1$ (limit equidistribution).

2010 Mathematics Subject Cassification: Lehmer conjecture, Schinzel-Zassenhaus conjecture, Mahler measure, minoration, Dobrowolski inequality, asymptotic expansion, transfer operator, dynamical zeta function, Rényi-Parry β-shift, Parry Upper function, Perron number, Pisot number, Salem number, Parry number, totally real algebraic numbers, limit equidistribution.
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Contents

1 Introduction

The question asked by Lehmer in [Le] (1933) about the existence of integer univariate polynomials of Mahler measure arbitrarily close to one became a conjecture. Lehmer's Conjecture is stated as follows: there exists an universal constant $c > 0$ such that the Mahler measure $\mathrm{M}(\alpha)$ satisfies $\mathrm{M}(\alpha) \geq 1 + c$ for all nonzero algebraic numbers α, not being a root of unity (Amoroso [A2] [A3], Blansky and Montgomery [ByM], Boyd [Bo4] [Bo5], Cantor and Strauss [CS], Dobrowolski [Do2], Dubickas [Ds2], Hindry and Silverman [HyS], Langevin [Lg], Laurent [La], Louboutin [Lt], Masser [Mr4], Mossinghoff, Rhin and Wu [MRW], Schinzel [Sc2], Silverman [Sn], Smyth [Sy5] [Sy6], Stewart [St], Waldschmidt [W] [W2]).

If α is a nonzero algebraic integer, $\mathrm{M}(\alpha) = 1$ if and only if $\alpha = 1$ or is a root of unity by Kronecker's Theorem (1857) [Krr]. Lehmer's Conjecture asserts a discontinuity of the value of $\mathrm{M}(\alpha)$, $\alpha \in \mathscr{O}_{\overline{\mathbb{Q}}}$, at 1. In § 2 we evoke the meaning of this

discontinuity in different contexts, in particular in number theory following Bombieri [Bri], Dubickas [Ds9] and Smyth [Sy6].

In this note we prove that Lehmer's Conjecture is true by establishing minorations of the Mahler measure $\mathrm{M}(\alpha)$ for any nonzero algebraic integer α which is not a root of unity. The proof contains a certain number of new notions which call for comments, and basically brings the dynamical zeta functions $\zeta_\beta(z)$ of the β-shift associated to algebraic numbers, their poles and their Poincaré asymptotic expansions, into play [VG7]. Let us describe briefly the ingredients which will appear in the proof. Due to the invariance of the Mahler measure $\mathrm{M}(\alpha)$ by the transformations $z \to \pm z^{\pm 1}$ and $z \to \pm \overline{z}^{\pm 1}$, it is sufficient to consider the two following cases:

(i) α real algebraic integer > 1, in which case α is generically named β,

(ii) α nonreal complex algebraic integer, $|\alpha| > 1$, with $\arg(\alpha) \in (0, \pi/2]$,

with $|\alpha| > 1$ sufficiently close to 1 in both cases. In Section § 6 we show how the nonreal complex case **(ii)** can be deduced from the real case **(i)** by considering the Rényi-Parry dynamics of the houses $\lceil\overline{\alpha}\rceil$. Now, in case **(i)**, by the Northcott property, the degree $\deg(\beta)$, valued in $\mathbb{N} \setminus \{0,1\}$, is necessarily not bounded when $\beta > 1$ tends to 1^+. To compensate the absence of an integer function of β which "measures" the proximity of β with 1, we introduce the natural integer function of β, that we call the *dynamical degree of* β, denoted by $\mathrm{dyg}(\beta)$, which is defined by the relation: for $1 < \beta \leq \frac{1+\sqrt{5}}{2}$ any real number, $\mathrm{dyg}(\beta)$ is the unique integer $n \geq 3$ such that

$$\theta_n^{-1} \leq \beta < \theta_{n-1}^{-1} \tag{1.1}$$

where θ_n is the unique root in $(0,1)$ of the trinomial $G_n(z) = -1 + z + z^n$. The (unique) simple zero > 1 of the trinomial $G_n^*(z) := 1 + z^{n-1} - z^n, n \geq 2$, is the Perron number θ_n^{-1}. The set of dominant roots $(\theta_n^{-1})_{n\geq 2}$ of the nonreciprocal trinomials $(G_n^*(z))_{n\geq 2}$ constitute a strictly decreasing sequence of Perron numbers, tending to one. Section § 3 summarizes the properties of these trinomials. The sequence $(\theta_n^{-1})_{n\geq 2}$ will be extensively used in the sequel. It is a fundamental set of Perron numbers of the interval $(1, \theta_2^{-1})$ simply indexed by the integer n, and this indextion is extended to any real number β lying between two successive Perron numbers of this family by (1.1). Let us note that $\mathrm{dyg}(\beta)$ is well-defined for algebraic integers β and also for transcendental numbers β. Let $\kappa := \kappa(1, a_{\max}) = 0.171573\ldots$ (cf below, and §3 §5.1 §6.5 for the proofs).

Theorem 1.1. (i) *For* $n \geq 2$,

$$\mathrm{dyg}(\theta_n^{-1}) = n = \begin{cases} \deg(\theta_n^{-1}) & \text{if} \quad n \not\equiv 5 \ (\mathrm{mod}\, 6), \\ \deg(\theta_n^{-1}) + 2 & \text{if} \quad n \equiv 5 \ (\mathrm{mod}\, 6), \end{cases} \tag{1.2}$$

(ii) *if β is a real number which satisfies $\theta_n^{-1} \leq \beta < \theta_{n-1}^{-1}$, $n \geq 2$, then the asymptotic expansion of the dynamical degree* $\mathrm{dyg}(\beta) = \mathrm{dyg}(\theta_n^{-1}) = n$ *of β is:*

$$\mathrm{dyg}(\beta) = -\frac{\mathrm{Log}\,(\beta - 1)}{\beta - 1}\Big[1 + O\Big(\Big(\frac{\mathrm{Log}\,(-\mathrm{Log}\,(\beta-1))}{\mathrm{Log}\,(\beta-1)}\Big)^2\Big)\Big], \tag{1.3}$$

with the constant 1 *in the Big O; moreover, if* $\beta \in (\theta_n^{-1}, \theta_{n-1}^{-1})$, $n \geq 260$, *is an algebraic integer of degree* $\deg(\beta)$, *then*

$$\mathrm{dyg}(\beta)\Big(\frac{2\arcsin\big(\frac{\kappa}{2}\big)}{\pi}\Big) + \Big(\frac{2\kappa \operatorname{Log}\kappa}{\pi\sqrt{4-\kappa^2}}\Big) \;\leq\; \deg(\beta). \tag{1.4}$$

Moreover, for α any nonreal complex algebraic integer, $|\alpha| > 1$, such that $1 < \lceil\alpha\rceil \leq \frac{1+\sqrt{5}}{2}$, the dynamical degree of α is defined by $\mathrm{dyg}(\alpha) := \mathrm{dyg}(\lceil\alpha\rceil)$; by extension, if $P(X)$ is an irreducible integer monic polynomial, we define $\mathrm{dyg}(P)$ as $:= \mathrm{dyg}(\lceil\alpha\rceil)$ for any root α of P.

In [VG6], the problem of Lehmer for the family $(\theta_n^{-1})_{n\geq 2}$ was solved using the Poincaré asymptotic expansions of the roots of (G_n) of modulus < 1 and of the Mahler measures $(\mathrm{M}(\theta_n^{-1}))_{n\geq 2}$. The purpose of the present note is to extend this method to any algebraic integer β of dynamical degree $\mathrm{dyg}(\beta)$ large enough, to show that this method allows to prove that the Conjecture of Lehmer is true in general.

The choice of the sequence of trinomials (G_n) is fairly natural in the context of the Rényi-Parry dynamical systems (recalled in § 4) and leads to a theory of perturbation of these trinomials compatible with the dynamics (in § 4.5). Therefore, in the present dynamical approach, taking the integer function $\mathrm{dyg}(\beta)$ as an integer variable tending to infinity when $\beta > 1$ tends to 1^+ is natural. All the asymptotic expansions, for the roots of modulus < 1 of the minimal polynomials $P_\beta(z)$, for the lower bounds of the lenticular Mahler measures $\mathrm{M}_r(\beta)$, will be obtained as a function of the integer $\mathrm{dyg}(\beta)$, when β tends to 1^+.

To the β-shift, to the Rényi-Parry dynamical system associated with an algebraic integer $\beta > 1$ are attached several analytic functions: (i) first, the minimal polynomial function $P_\beta(z)$ which is (monic) reciprocal by a Theorem of C. Smyth [Sy] as soon as $\mathrm{M}(\beta) < \Theta = 1.3247\ldots$; (ii) then the (Artin-Mazur) dynamical zeta function of the β-shift [AMr], the generalized Fredholm determinant of the transfer operator associated with the β-transformation T_β [BaK], the Perron-Frobenius operator associated to T_β [IT] [Mo] [Mo2] [T], the kneading determinants, and its variants, of the kneading theory of Milnor and Thurston [MorT]. Their are closely related and recalled in § 4. Finally the main theorems below will be obtained using the Parry Upper function $f_\beta(z)$, constructed from the inverse of the dynamical zeta function $\zeta_\beta(z)$. The Parry Upper function is a generalization of the Fredholm determinant associated with the transfer operator of the β-transformation.

Using ergodic theory Takahaski [T] [T2], Ito and Takahashi [IT], Flatto, Lagarias and Poonen [FLP] gave an explicit expression of the Parry Upper function $f_\beta(z)$ as a function of the dynamical zeta function $\zeta_\beta(z)$ of the β-shift. These expressions (§4.2) will be extensively used in the sequel.

The Parry Upper upper function at β takes the general form, with a lacunarity controlled by the dynamical degree (Theorem 4.6):

$$f_\beta(z) = -1 + z + z^{\mathrm{dyg}(\beta)} + z^{m_1} + z^{m_2} + z^{m_3} + \ldots$$

$$= G_{\mathrm{dyg}(\beta)} + z^{m_1} + z^{m_2} + z^{m_3} + \ldots \tag{1.5}$$

with $m_1 \geq 2\,\mathrm{dyg}(\beta) - 1, m_{q+1} - m_q \geq \mathrm{dyg}(\beta) - 1, q \geq 1$. For $\theta_{\mathrm{dyg}(\beta)}^{-1} \leq \beta < \theta_{\mathrm{dyg}(\beta)-1}^{-1}$, the lenticulus $\mathscr{L}_\beta$ of zeroes of $f_\beta(z)$ relevant for the Mahler measure is obtained by a deformation of the lenticulus of zeroes $\mathscr{L}_{\theta_{\mathrm{dyg}(\beta)}^{-1}}$ of $G_{\mathrm{dyg}(\beta)}$ due to the tail $z^{m_1} + z^{m_2} + \ldots$ itself. For instance, for $\beta = 1.17628\ldots$ Lehmer's number (Table 1), $\mathrm{dyg}(\beta) = 12$,

$$f_\beta(z) = -1 + z + z^{12} + z^{31} + z^{44} + z^{63} + z^{86} + z^{105} + z^{118} + \ldots$$

is sparse with gaps of length $\geq 10 = \mathrm{dyg}(\beta) - 2$ and $\mathscr{L}_\beta$ is close to $\mathscr{L}_{\theta_{12}^{-1}}$ (Fig. 1).

The passage from the Parry Upper function $f_\beta(z)$ to the Mahler measure $\mathrm{M}(\beta)$ (when $\beta > 1$ is an algebraic integer) is crucial, constitutes the main discoveries of the author, and relies upon two facts: (i) the <u>discovery of lenticular distributions of zeroes</u> of $f_\beta(z)$ in the annular region $e^{-\mathrm{Log}\,\beta} = \frac{1}{\beta} \leq |z| < 1$ which are very close to the lenticular sets of zeroes of the trinomials $G_{\mathrm{dyg}(\beta)}(z)$ of modulus < 1; (ii) <u>the identification of these zeroes as conjugates of β</u>. The quantity $\mathrm{Log}\,\beta$ is the topological entropy of the β-shift. These lenticular distributions of zeroes lie in the cusp of the fractal of Solomyak of the β-shift [Sk] (recalled in § 4.2.2). The key ingredient for obtaining the Dobrowolski type minoration of the Mahler measure $\mathrm{M}(\beta)$ in Theorem 1.4 relies upon the best possible evaluation of the deformation of these lenticuli of zeroes by the method of Rouché (in § 5) and the coupling between the Rouché conditions and the asymptotic expansions of the lenticular zeroes.

The identification of the complete set of conjugates of β ($\beta > 1$ being an algebraic integer) seems to be unreachable by this method. Only lenticular conjugates of modulus < 1 can be identified in an angular subsector of $\arg(z) \in [-\frac{\pi}{3}, \frac{\pi}{3}]$ (Theorem 5.23 and Theorem 6.1). Consequently the present method only gives access to a "*part*" of the Mahler measure itself. We denote by $\mathscr{L}_\beta, \mathscr{L}_{\theta_{\mathrm{dyg}(\beta)}^{-1}}$ the lenticular sets of zeroes of $f_\beta(z)$, resp. of $G_{\mathrm{dyg}(\beta)}(z)$. We call

$$\mathrm{M}_r(\beta) = \prod_{\omega \in \mathscr{L}_\beta} |\omega|^{-1}$$

the lenticular Mahler measure of β. It satisfies $\mathrm{M}_r(\beta) \leq \mathrm{M}(\beta)$.

We show that $\beta \to \mathrm{M}_r(\beta)$ is continuous on each open interval $(\theta_n^{-1}, \theta_{n-1}^{-1})$ for the usual topology, and that it admits a lower bound which can be expanded as an asymptotic expansion of $\mathrm{dyg}(\beta)$ (Theorem 6.5). The general minorant of $\mathrm{M}(\beta)$ we are looking for to solve the problem of Lehmer comes from the asymptotic expansion of the lower bound of $\mathrm{M}_r(\beta)$, as in (1.20).

Given a nonzero algebraic integer α which is not a root of unity, with $|\alpha|$ close enough to 1^+, two new general notions appear in the present study:

(i) the *continuity* properties of the lenticular Mahler measure with the house of α (Theorem 5.24 and Remark 5.25) and the importance of the *dynamics* of $\lceil\overline{\alpha}\rceil$ in the identification of the conjugates,

(ii) the canonical *fracturability* of the minimal polynomial $P_\alpha(z)$ as a product of two integer (arithmetic) series whose one is the Parry Upper function $f_\beta(z)$ at β (Theorem 5.23 and Theorem 6.1).

This *cleavability* of the irreducible elements $P_\alpha(z)$ which are the minimal polynomials of the algebraic integers α close to 1^+ in modulus, obeying the *Carlson-Polya dichotomy* [C] [Pl] in their canonical decomposition (cf Theorem 5.23, Theorem 6.1 for the definitions), as

$$P_\beta(z) = U_\beta(z) \times f_\beta(z) \qquad \begin{cases} \text{on } \mathbb{C} & \text{if } \beta \text{ is a Parry number, with} \\ & U_\beta \text{ and } f_\beta \text{ as meromorphic functions,} \\ & \\ \text{on } |z| < 1 & \text{if } \beta \text{ is a nonParry number, with } |z| = 1 \\ & \text{as natural boundary for both } U_\beta \text{ and } f_\beta, \end{cases} \tag{1.6}$$

seems to be new. It will probably call for a refoundation of the theory of divisibility in Commutative Algebra based on the theory of the arithmetic power series coming from the dynamics, from the positive, negative [N] or generalized β-shift [Ga] [Th], which are the Parry Upper functions. In the present note we only use the (positive) β-shift.

Our main theorems are the following.

Theorem 1.2 (ex-Lehmer conjecture). *For any nonzero algebraic integer α which is not a root of unity,*

$$\mathrm{M}(\alpha) \geq \theta_{259}^{-1} = 1.016126\ldots$$

In terms of the Weil height h, this minoration is restated as:

$$h(\alpha) \geq \frac{\mathrm{Log}\,(\theta_{259}^{-1})}{\deg(\alpha)}. \tag{1.7}$$

Theorem 1.3 (ex-Schinzel-Zassenhaus conjecture). *Let α be a nonzero algebraic integer which is not a root of unity. Then*

$$\lceil\overline{\alpha}\rceil \geq 1 + \frac{c}{\deg(\alpha)} \tag{1.8}$$

with $c = \theta_{259}^{-1} - 1 = 0.016126\ldots$.

The following definitions are given in § 5. We just report them here for stating Theorem 1.4. Denote by $a_{\max} = 5.87433\ldots$ the abscissa of the maximum of the function $a \to \kappa(1,a) := \frac{1-\exp(\frac{-\pi}{a})}{2\exp(\frac{\pi}{a})-1}$ on $(0,\infty)$ (Figure 2). Let $\kappa := \kappa(1, a_{\max}) = 0.171573\ldots$ be the value of the maximum. Let $S := 2\arcsin(\kappa/2) = 0.171784\ldots$. Denote

$$\Lambda_r\mu_r := \exp\Big(\frac{-1}{\pi}\int_0^S \mathrm{Log}\Big[\frac{1+2\sin(\frac{x}{2}) - \sqrt{1-12\sin(\frac{x}{2}) + 4(\sin(\frac{x}{2}))^2}}{4}\Big]dx\Big)$$

$$= 1.15411\ldots, \qquad \text{a value slightly below Lehmer's number } 1.17628\ldots \tag{1.9}$$

Recall that Lehmer's number is the smallest Mahler measure (> 1) of algebraic integers known and the smallest Salem number known [Mlist] [MRW], dominant root of the degree 10 Lehmer's polynomial

$$X^{10} + X^9 - X^7 - X^6 - X^5 - X^4 - X^3 + X + 1. \tag{1.10}$$

Theorem 1.4 (Dobrowolski type minoration). *Let α be a nonzero algebraic integer which is not a root of unity such that* $\mathrm{dyg}(\alpha) \geq 260$. *Then*

$$\mathrm{M}(\alpha) \geq \Lambda_r \mu_r - \Lambda_r \mu_r \frac{S}{2\pi} \Big(\frac{1}{\mathrm{Log}\,(\mathrm{dyg}(\alpha))}\Big) \tag{1.11}$$

In terms of the Weil height h, using Theorem 5.3, the asymptotics of the minoration (1.11) takes the following form:

$$\deg(\alpha) h(\alpha) \;\geq\; \mathrm{Log}\,(\Lambda_r \mu_r) + \frac{S}{2\pi} \frac{1}{\mathrm{Log}\,(\lceil\alpha\rceil - 1)}. \tag{1.12}$$

The minoration (1.11) can also be restated in terms of the usual degree. Let $B > 0$. Let us consider the subset $\mathscr{F}_B$ of all nonzero algebraic integers α not being a root of unity such that $\lceil\alpha\rceil < \theta_{259}^{-1}$ satisfying $n := \deg(\alpha) \leq (\mathrm{dyg}(\alpha))^B$. Then

$$\mathrm{M}(\alpha) \geq \Lambda_r \mu_r - \Lambda_r \mu_r \frac{SB}{2\pi} \Big(\frac{1}{\mathrm{Log}\, n}\Big), \qquad \alpha \in \mathscr{F}_B. \tag{1.13}$$

Comparatively, in 1979, Dobrowolski [Do2], using an auxiliary function, obtained the well-known asymptotic minoration

$$\mathrm{M}(\alpha) > 1 + (1 - \varepsilon) \left(\frac{\mathrm{Log}\,\mathrm{Log}\, n}{\mathrm{Log}\, n}\right)^3, \qquad n > n_0, \tag{1.14}$$

with $1 - \varepsilon$ replaced by $1/1200$ for $n \geq 2$, for an effective version of the minoration. Here, in (1.11) or (1.13), the constant in the minorant is not any more 1 but $1.15411\ldots$ and the sign of the n-dependent term becomes negative, with an appreciable gain of $(\mathrm{Log}\, n)^2$ in the denominator.

The minoration (1.11) is general and admits a much better lower bound, in a similar formulation, when α only runs over the set of Perron numbers $(\theta_n^{-1})_{n\geq 2}$. Indeed, in [VG6] , it is shown that

$$\mathrm{M}(\theta_n^{-1}) \;>\; \Lambda - \frac{\Lambda}{6} \Big(\frac{1}{\mathrm{Log}\, n}\Big), \qquad n \geq 2, \tag{1.15}$$

holds with the following constant of the minorant

$$\Lambda := \exp\big(\frac{3\sqrt{3}}{4\pi} \mathrm{L}(2, \chi_3)\big) \;=\; \exp\Big(\frac{-1}{\pi} \int_0^{\pi/3} \mathrm{Log}\,\big(2 \sin\big(\frac{x}{2}\big)\big) dx\Big) \;=\; 1.38135\ldots, \tag{1.16}$$

much higher than 1 and $1.1541\ldots$, and $L(s,\chi_3) := \sum_{m\geq 1} \frac{\chi_3(m)}{m^s}$ the Dirichlet L-series for the character χ_3, with χ_3 the uniquely specified odd character of conductor 3 ($\chi_3(m) = 0, 1$ or -1 according to whether $m \equiv 0$, 1 or 2 (mod 3), equivalently $\chi_3(m) = \left(\frac{m}{3}\right)$ the Jacobi symbol). Numerically the constant $\Lambda_r \mu_r S/(2\pi) = 0.0315536\ldots$ in (1.11) is much smaller than $\Lambda/6 = 0.230225\ldots$ in (1.15). After Smyth's Conjecture 1.1., quoted in Flammang in [Fg], on height one integer trinomials, the minoration (1.15) admits the following conjectural generalization, making sense, with a constant term in the minorant also equal to Λ.

Conjecture 1. *There exists a constant* $\nu > 0$ *such that, for* $1 \leq k < n/2$, $\gcd(n,k) = 1$,

$$\mathrm{M}(X^n \pm X^k \pm 1) \; > \; \Lambda - \nu \left(\frac{1}{\mathrm{Log}\, n}\right), \qquad n \geq 2. \tag{1.17}$$

In the sense of Boyd-Lawton's limit Theorem [Bo7] [Lw] $\mathrm{M}(X^n \pm X^k \pm 1)$ is close to the bivariate Mahler measure $\mathrm{M}(y \pm x \pm 1)$. Links between values of Dirichlet L-series $L'(\chi,s)$ (χ character) at algebraic numbers and Mahler measures of multivariate integer polynomials were discovered by Smyth [Sy5] in 1981, together with some asymptotic formulas, followed by Ray [Ry]. The minoration (1.17) would strongly improve the one deduced from [Sy5] or from Boyd-Lawton's Theorem completed by the analytic asymptotics of Condon [Cdn] [Cdn2] in terms of polylogarithms.

Given an integer monic irreducible univariate polynomial $P(X)$ for which there exists an integer d-variate polynomial $Q(X_1, X_2, \ldots, X_d)$ such that $\mathrm{M}(P)$ is close to $\mathrm{M}(Q)$ in the sense of Boyd-Lawton's Theorem [Cdn] [Cdn2] [Et], then, from Theorem 1.4 and Conjecture 1, the following minoration is conjecturally expected

$$\mathrm{M}(P) > \nu_0 - \nu\left(\frac{1}{\mathrm{Log}\,(\mathrm{dyg}(P))}\right) \tag{1.18}$$

where $\nu_0 = \mathrm{M}(Q)$ and, in which, if $\mathrm{dyg}(P)$ is replaced by $\deg(P)$, ν depends upon the "limit polynomial" Q. What would be the set of possible constants $\{\nu_0\}$? This set belongs to the (mostly conjectural) world of special values of L-functions. After Deninger [Dgr] showed in 1997, when Q does not vanish on $\mathbb{T}^d$, how to interpret the logarithmic Mahler measures $\mathrm{Log}\, M(Q)$ as Deligne periods of mixed motives, dozans of formulas were guessed by Boyd [Bo16], many remaining unproved yet [BoRs] [Lin] [Lin2] [Rgs] [ShVo] [Zun]; Boyd's formulas link these logarithmic Mahler measures to sums of special values of different L-functions and their derivatives, i.e. to Dirichlet L-series $L(\chi,s)$ (χ character) at algebraic numbers, values of Hasse-Weil L-funtions $L(E,s)$ of elliptic curves $E/\mathbb{Q}$, etc. Rodriguez-Villegas in [RVs] showed how to correlate Boyd's formulas to the Bloch-Beilinson's conjectures, investigating the domain of applicability of these conjectures. Bornhorn [Brn], Standfest [Sst] continued the motivic approach of Deninger to interpret Boyd's formulas in the light of Beilinson's conjectures. In the same direction Lalín [Lin] [Lin2] continued developping the ideas of Deninger towards polylogarithmic motivic complexes using Bloch

groups, a motivic cohomology of algebraic varieties and Beilinson's regulator, in the multivariable case.

The Mahler measure $\mathrm{M}(G_n)$ of the trinomial G_n is equal to the lenticular Mahler measure $\mathrm{M}_r(G_n)$ itself, with limit $\lim_{n\to+\infty}\mathrm{M}(G_n) = \lim_{n\to+\infty}\mathrm{M}_r(G_n) = \Lambda$, having asymptotic expansion

$$\mathrm{M}(G_n) \;=\; \Lambda\Big(1 + r(n)\,\frac{1}{\mathrm{Log}\,n} + O\Big(\frac{\mathrm{Log}\,\mathrm{Log}\,n}{\mathrm{Log}\,n}\Big)^2\Big) \tag{1.19}$$

with $r(n)$ real, $|r(n)| \leq 1/6$. In the case of the trinomials G_n the characterization of the roots of modulus < 1 is direct (§3) and does not require the detection method of Rouché. In the general case, with $\beta \in (\theta_n^{-1}, \theta_{n-1}^{-1})$, $n = \mathrm{dyg}(\beta)$ large enough, applying the method of Rouché only leads to the following asymptotic lower bound of the lenticular minorant, similarly as in (1.19), as (§6.2):

$$\mathrm{M}_r(\beta) \geq \Lambda_r\mu_r\big(1 + \frac{\mathscr{R}}{\mathrm{Log}\,n} + O\big(\big(\frac{\mathrm{Log}\,\mathrm{Log}\,n}{\mathrm{Log}\,n}\big)^2\big), \quad \text{with } |\mathscr{R}| < \frac{\arcsin(\kappa/2)}{\pi}. \tag{1.20}$$

Denote by $M_{\inf} := \liminf_{\lceil\alpha\rceil \to 1^+}\mathrm{M}(\alpha)$ the limit infimum of the Mahler measures $\mathrm{M}(\alpha)$, $\alpha \in \mathscr{O}_{\overline{\mathbb{Q}}}$, when $\lceil\alpha\rceil > 1$ tends to 1^+. Then

$$\Lambda_r\mu_r \leq M_{\inf} \leq \Lambda. \tag{1.21}$$

Whatever the expression of the constant terms ν_0 as sums of special values of different L-functions, because the β-shift used in the present approach for the dynamization of the algebraic equations (§ 4) is compact, we formulate a continuum for the set of the constant terms of the minorants in (1.18), as follows.

Conjecture 2. *For any $\nu_0 \in [M_{\inf}, \Lambda)$ there exists a sequence of integer monic irreducible polynomials $(H_m(z))_m$ such that* $\lim_{m\to+\infty}\mathrm{M}(H_m) = \nu_0$.

Lenticuli of conjugates lie in the cusp of Solomyak's fractal (§ 4.2.2). The number of elements of a lenticulus $\mathscr{L}_\alpha$ is an increasing function of the dynamical degree $\mathrm{dyg}(\alpha)$ as soon as $\mathrm{dyg}(\alpha)$ is large enough. The existence of lenticuli composed of three elements only (one real, a pair of nonreal complex-conjugated conjugates) is studied carefully in § 7.1. Such lenticuli appear at small dynamical degrees. Since Salem numbers have no nonreal complex conjugate of modulus < 1, they should not possess 3-elements lenticuli of conjugates, therefore they should possess a small dynamical degree bounded from above. We obtain 31 as an upper bound as follows.

Theorem 1.5 (ex-Lehmer conjecture for Salem numbers). *Let T denote the set of Salem numbers. Then T is bounded from below:*

$$\beta \in T \qquad \Longrightarrow \qquad \beta > \theta_{31}^{-1} = 1.08544\ldots$$

Lehmer's number $1.17628\ldots$ belongs to the interval $(\theta_{12}^{-1}, \theta_{11}^{-1})$ (Table 1). This interval does not contain any other known Salem number. If there is another one, its degree should be greater than 44 [Mf] [MRW].

Conjecture 3. *There is no Salem number in the interval*

$$(\theta_{31}^{-1}, \theta_{12}^{-1}) = (1.08544\ldots, 1.17295\ldots).$$

Parry Upper functions $f_\beta(z)$, with β being an algebraic integer of dynamical degree $\mathrm{dyg}(\beta) = 12$ to 16, do possess 3-elements lenticuli of zeroes in the open unit disc (as in Fig. 1). Conjecture 1 means that there should exist a method better than Rouché's method to identify these 3-elements lenticuli of zeroes as 3-elements lenticuli of conjugates of β.

Our results give another proof that the field of totally real algebraic numbers $\mathbb{Q}^{tr}$ has the Bogomolov property relative to the Weil height h (§ 7.3), as follows.

Theorem 1.6. *Let $\mathbb{Q}^{tr}$ denote the field of all totally real algebraic numbers. If h denotes the absolute logarithmic Weil height,*

$$\alpha \in \mathbb{Q}^{tr}, \alpha \neq 0, \neq \pm 1 \quad \Rightarrow \quad h(\alpha) > \frac{1}{4} \operatorname{Log} \theta_{31}^{-1} = 0.020498\ldots \tag{1.22}$$

Since Fili and Miner [FMr3] [FMr4] proved that

$$\liminf_{\alpha \in \mathbb{Q}^{tr}, \alpha \neq 0, \pm 1} h(\alpha) > 0.120786\ldots,$$

the interval $(0.020498, 0.120786)$ only contains a finite number of values of $h(\alpha)$ (isolated values). Eventually this number is zero. What are they? What are the corresponding totally real algebraic numbers α?

Though Lehmer's problem arises from lenticuli of conjugates in angular sectors containing 1, the complete set of conjugates equidistribute on the unit circle at the limit, once the Mahler measures are small enough, as follows.

Theorem 1.7. *Let $(\alpha_q)_{q \geq 1}$ be a sequence of nonzero algebraic integers which are not roots of unity such that $\mathrm{M}(\alpha_q) < \Theta$, $\mathrm{dyg}(\alpha_q) \geq 260$ for $q \geq 1$, with $\lim_{q \to +\infty} \boxed{\alpha_q} = 1$. Then the sequence $(\alpha_q)_{q \geq 1}$ is strict and*

$$\mu_{\alpha_q} \ \to \ \mu_{\mathbb{T}}, \qquad \mathrm{dyg}(\alpha_q) \to +\infty, \qquad \textit{weakly}, \tag{1.23}$$

i.e. for all bounded, continuous functions $f : \mathbb{C}^\times \to \mathbb{C}$,

$$\int f d\mu_{\alpha_q} \to \int f d\mu_{\mathbb{T}}, \qquad \mathrm{dyg}(\alpha_q) \to +\infty. \tag{1.24}$$

Theorem 1.7 can also be considered as an important theorem for the understanding of the limit distribution of roots of families of random polynomials, of small Mahler measure, [HNi] [SiYv].

Following work of Langevin [Lg] [Lg2] [Lg3] [Mt3] the importance of the angular sectors containing the point 1 was already guessed by Dubickas and Smyth [DsSy], Rhin, Smyth and Wu [RnS] [RnW]. The lenticular probability distribution of zeroes of the Parry Upper functions $f_\beta(z)$ in this sector, which are identified as Galois

conjugates of β, admits the uniform limit $\mu_{\mathbb{T},arc}$ which denotes the restriction of the (normalized) Haar measure $\mu_{\mathbb{T}}$ to the limit arc (for the weak topology)

$$\{z \mid |z| = 1, \arg(z) \in [-2\arcsin\left(\tfrac{\kappa}{2}\right), +2\arcsin\left(\tfrac{\kappa}{2}\right)]\} \text{ on the unit circle}$$

(Proposition 5.12, Remark 5.13 (i), Theorem 5.15, [VG6] Theorem 6.2).

The very nature of the set of Parry numbers (Definition 4.1) is a deep question addressed to the dynamics of Perron numbers (Adler and Marcus [AM], Bertrand-Mathis [BMs], Boyd [Bo7], Boyle and Handelman [Ble2] [BleH] [BleH2], Brunotte [Bte], Calegari and Huang [CiHg], Dubickas and Sha [DsSha], Lind [Ld] [Ld2], Lind and Marcus [LdM], Thurston [Tn2], Verger-Gaugry [VG2] [VG3]), associated with the rationality of the dynamical zeta function of the β-shift

$$\zeta_\beta(z) := \exp\left(\sum_{n=1}^{\infty} \frac{\mathscr{P}_n}{n} z^n\right), \quad \mathscr{P}_n := \#\{x \in [0,1] \mid T_\beta^n(x) = x\} \tag{1.25}$$

counting the number of periodic points of period dividing n (isolated points). For α a nonzero algebraic integer which is not a root of unity, with $\beta = \lceil \alpha \rceil$, by Theorem 4.12,

$$\beta \quad \text{is a Parry number} \quad \text{iff} \quad \zeta_\beta(z) \quad \text{is a rational function;}$$

and $|z| = 1$ is the natural boundary of the domain of fracturability of the minimal polynomial P_α if and only if β is not a Parry number (Theorem 6.1), as soon as $|\alpha|$ is close enough to 1, in the Carlson-Polya dichotomy. Comparatively, for complete nonsingular projective algebraic varieties X over the field of q elements, q a prime power, the zeta function $\zeta_X(t)$ introduced by Weil [We] is only a rational function (Dwork [Dwk], Tao [Tao]): the first Weil's conjecture, for which there exists a set of characteristic values was proved by Dwork using p-adic methods (Dwork [Dwk]), and "Weil II", the Riemann hypothesis, proved by Deligne using l-adic étale cohomology in characteristic $p \neq l$ (Deligne [Dne]). It is defined as a dynamical zeta function with the action of the Frobenius. The purely p-adic methods of Dwork (Dwork [Dwk]), continued by Kedlaya [Kya] for "Weil II" in the need of numerically computing zeta functions by explicit equations, allow an intrinsic computability, as in Lauder and Wan [LrWn], towards a p-adic cohomology theory, are linked to "extrinsic geometry", to the defining equations of the variety itself. They are in contrast with the relative version of crystalline cohomology developped by Faltings [Fgs], or the Monsky-Washnitzer constructions used by Lubkin [Lkn]. We refer the reader to Robba [Rba], Kedlaya [Kya], Tao [Tao], for a short survey on other developments.

After Weil [We], and introduced in general terms by Artin and Mazur in [AMr], the theory of dynamical zeta functions $\zeta(z)$ associated with dynamical systems, based on an analogy with the number theory zeta functions, developped under the impulsion of Ruelle [Ru4] in the direction of the thermodynamic formalism and with Pollicott, Baladi and Keller [BaK] towards transfer operators and counting orbits [PaPt]. The

determination and the existence of meromorphic extensions or/and natural boundaries of dynamical zeta functions is a deep problem.

In the present proof of the conjecture of Lehmer, the analytic extension of the dynamical zeta function of the β-shift behaves as an analogue of Weil's zeta function (in the sense that both are dynamical zeta functions). But it generates questions beyond the analogues of Weil's conjectures since not only the rational case of ζ_β contribute to the minoration of the Mahler measure, but also the nonrational case with the unit circle as natural boundary and lenticular poles arbitrarily close to it. The equivalent of "Weil II" (Riemann Hypothesis) would be the determination of the geometry of the beta-conjugates in the rationality case. Beta-conjugates are zeroes of Parry polynomials, whose factorization was studied in the context of the theory of Pinner and Vaaler [PrVr] in [VG3].

An apparent difficulty for the computation of the minorant of $\mathrm{M}(\alpha)$ comes from the absence of complete characterization of the set of Parry numbers $\mathbb{P}_P$, when $\beta = \lceil\overline{\alpha}\rceil$ is close to one, since we never know whether β is a Parry number or not. But the Mahler measure $\mathrm{M}(\alpha)$ is independent of the Carlson-Polya dichotomy. Indeed, the two domains of definitions of ζ_β, "$\mathbb{C}$" and "$|z| < 1$", together with the corresponding splitting (1.6), may occur fairly "randomly" when β tends to one. But $\{|z| < 1\}$ is a domain included in both, $\mathrm{M}(\alpha)$ "reading" only the roots in it and not taking care of the "status" of the unit circle. Whether $f_\beta(z)$ can be continued analytically or not beyond the unit circle has no effect on the value of the Mahler measure $\mathrm{M}(\alpha)$.

Many consequences of the above Theorems can be readily deduced. Let us mention a few of them below and in § 9. A first consequence concerns the theory of heights on $\overline{\mathbb{Q}}^\times$ written multiplicatively [AkV]: the metric, t-metric and ultrametric Mahler measures [DsSy] [Sls] [FSls] induce the *discrete topology on* $\overline{\mathbb{Q}}^\times/\mathrm{Tor}(\overline{\mathbb{Q}}^\times)$, with its consequences.

A second consequence concerns the difference between two successive Salem numbers. In the context of root separation theorems [BBG] [BgMte] [Eve] [Gng] [Mhr2] and the representability of real algebraic numbers as a difference of two Mahler measures [DrDs] the difference between two successive Salem numbers of the same degree (in particular when the degree is very large) admits the following universal lower bound, readily deduced from Lemma 4 in [Sy6] and Theorem 1.5.

Theorem 1.8. *Let $d \geq 4$ be an integer. Denote by $T_{(d)}$ the subset of T of the Salem numbers of degree d. Then,*

$$\tau, \tau' \in T_{(d)}, \quad \tau' > \tau \implies \tau' - \tau \geq \theta_{31}^{-1}(\theta_{31}^{-1} - 1) = 0.0927512\ldots$$

Distributions of algebraic numbers in very small neighbourhoods of Salem numbers can be studied by the algebraic coding with Stieltjes continued fractions [GdVG]. In higher dimension (cf §2.2), amongst other results, let us mention the following consequence which improves Laurent's Theorem 2.4 [La].

Theorem 1.9 (ex-Elliptic Lehmer Conjecture). *Let E/K be an elliptic curve over a number field K and $\widehat{h}$ the Néron-Tate height on $E(\overline{K})$. There is a positive constant $c(E/K)$ such that*

$$\widehat{h}(P) \geq \frac{c(E/K)}{[K(P):K]} \qquad \textit{for all } P \in E(\overline{K}) \setminus E_{\mathrm{tors}}. \tag{1.26}$$

The paper is self-contained and organized as follows: in §2 we adopt an interdisciplinary viewpoint to evoke the meaning of the Conjecture of Lehmer in different contexts together with the analogues of the Mahler measure and their geometrical counterparts. In §4 we give a semi-expository presentation of the analytic functions involved in the dynamics of the β-shift, $\beta > 1$, in particular of the Parry Upper function, useful for the sections §5 to §7. The proofs of Lehmer's Conjecture and Schinzel-Zassenhauss's Conjecture are obtained in §5 when β is a real algebraic integer > 1, and in §6. They are formulated in §6 in a general setting for any nonzero algebraic integer α which is not a root of unity, (i) which does not belong to $(1,\infty)$ or (ii) for which, if $\alpha > 1$, $\alpha \neq \lceil\overline{\alpha}\rceil =: \beta$. In §7 a Theorem of fracturability of the minimal integer polynomials of small Mahler measure is obtained which readily implies the Conjecture of Lehmer for Salem numbers, the Bogomolov property for totally real algebraic numbers. In §3 are presented the "à la Poincaré" technics of asymptotic expansions transposed (and adapted) to Number Theory from Celestial Mechanics; they were introduced for the first time in [VG6] on the height one trinomials G_n and G_n^*, their roots, and their Mahler measures. These asymptotic expansions are fundamental for dealing with the general cases of algebraic integers in §5, §6 and §7. Associated limit equidistribution results are obtained in §8. Theorems proving the equivalence of the Conjecture of Lehmer with some other statements in Geometry are gathered in §9.

Notations. Let $P(X) \in \mathbb{Z}[X]$, $m = \deg(P) \geq 1$. The *reciprocal polynomial* of $P(X)$ is $P^*(X) = X^m P(\frac{1}{X})$. The polynomial P is reciprocal if $P^*(X) = P(X)$. When it is monic, the polynomial P is said *unramified* if $|P(1)| = |P(-1)| = 1$. If $P(X) = a_0 \prod_{j=1}^{m}(X - \alpha_j) = a_0 X^m + a_1 X^{m-1} + \ldots + a_m$, with $a_i \in \mathbb{C}$, $a_0 a_m \neq 0$, and roots α_j, the *Mahler measure* of P is

$$\mathrm{M}(P) := |a_0| \prod_{j=1}^{m} \max\{1, |\alpha_j|\}. \tag{1.27}$$

The absolute Mahler measure of P is $\mathrm{M}(P)^{1/\deg(P)}$, denoted by $\mathscr{M}(P)$. The Mahler measure of an algebraic number α is the Mahler of its minimal polynomial P_α: $\mathrm{M}(\alpha) := \mathrm{M}(P_\alpha)$. For any algebraic number α the house $\lceil\overline{\alpha}\rceil$ of α is the maximum modulus of its conjugates, including α itself; by Jensen's formula the Weil height $h(\alpha)$ of α is $\mathrm{Log}\,\mathrm{M}(\alpha)/\deg(\alpha)$. By its very definition, $\mathrm{M}(PQ) = \mathrm{M}(P)\mathrm{M}(Q)$ (multiplicativity).

A *Perron number* is either 1 or a real algebraic integer $\theta > 1$ such that the Galois conjugates $\theta^{(i)}, i \neq 0$, of $\theta^{(0)} := \theta$ satisfy: $|\theta^{(i)}| < \theta$. Denote by $\mathbb{P}$ the set of Perron

numbers. A *Pisot number* is a Perron number > 1 for which $|\theta^{(i)}| < 1$ for all $i \neq 0$. The smallest Pisot number is denoted by $\Theta = 1.3247\ldots$, dominant root of $X^3 - X - 1$. A Salem number is an algebraic integer $\beta > 1$ such that its Galois conjugates $\beta^{(i)}$ satisfy: $|\beta^{(i)}| \leq 1$ for all $i = 1, 2, \ldots, m-1$, with $m = \deg(\beta) \geq 1$, $\beta^{(0)} = \beta$ and at least one conjugate $\beta^{(i)}, i \neq 0$, on the unit circle. All the Galois conjugates of a Salem number β lie on the unit circle, by pairs of complex conjugates, except $1/\beta$ which lies in the open interval $(0,1)$. Salem numbers are of even degree $m \geq 4$. The set of Pisot numbers, resp. Salem numbers, is denoted by S, resp. by T. If $\tau \in$ S or T, then $\mathrm{M}(\tau) = \tau$. A j-Salem number [Kda] [Set], $j \geq 1$, is an algebraic integer α such that $|\alpha| > 1$ and α has $j-1$ conjugate roots $\alpha^{(q)}$ different from α, satisfying $|\alpha^{(q)}| > 1$, while the other conjugate roots ω satisfy $|\omega| \leq 1$ and at least one of them is on the unit circle. We call the minimal polynomial of a j-Salem number a j-Salem polynomial. Salem numbers are 1-Salem numbers. A Salem number is said *unramified* if its minimal polynomial is unramified. We say that two Salem numbers λ and μ are *commensurable* if there exists positive integers k and l such that $\lambda^k = \mu^l$. Commensurability is an equivalence relation on T. Let $\lambda \in T$, K a subfield of $\mathbb{Q}(\lambda + \lambda^{-1})$, and $P_{\lambda,K}$ the minimal polynomial of λ over K; we say that λ is *square-rootable* over K if there exists a totally positive element $\alpha \in K$ and a monic reciprocal polynomial $q(x)$, whose even degree coefficients are in K and odd degree coefficients are in $\sqrt{\alpha}K$ such that $q(x)q(-x) = P_{\lambda,K}(x^2)$.

The set of algebraic numbers, resp. algebraic integers, in $\mathbb{C}$, is denoted by $\overline{\mathbb{Q}}$, resp. $\mathscr{O}_{\overline{\mathbb{Q}}}$. The nth cyclotomic polynomial is denoted by $\Phi_n(z)$. For any postive integer n, let $[n] := 1 + x + x^2 + \ldots + x^{n-1}$. The (naïve) height of a polynomial P is the maximum of the absolute value of the coefficients of P. Let A be a countable subset of the line; the *first derived set* $A^{(1)}$ of A is the set of the limit points of nonstationary infinite sequences of elements of A; the *k-th derived set* $A^{(k)}$ of A is the first derived set of $A^{(k-1)}$, $k \geq 2$.

For $x > 0$, $\lfloor x \rfloor$, $\{x\}$ and $\lceil x \rceil$ denotes respectively the integer part, resp. the fractional part, resp. the smallest integer greater than or equal to x. For $\beta > 1$ any real number, the map $T_\beta : [0,1] \to [0,1], x \to \{\beta x\}$ denotes the β-transformation. With $T_\beta^0 := T_\beta$, its iterates are denoted by $T_\beta^{(j)} := T_\beta(T_\beta^{j-1})$ for $j \geq 1$. A real number $\beta > 1$ is a Parry number if the sequence $(T_\beta^{(j)}(1))_{j \geq 1}$ is eventually periodic; a Parry number is called simple if in particular $T_\beta^{(q)}(1) = 0$ for some integer $q \geq 1$. The set of Parry numbers is denoted by $\mathbb{P}_P$. The terminology chosen by Parry in [Pa] has changed: β-numbers are now called Parry numbers, in honor of W. Parry.

The Mahler measure of a nonzero polynomial $P(x_1, \ldots, x_n) \in \mathbb{C}[x_1, \ldots, x_n]$ is defined by

$$\mathrm{M}(P) := \exp\left(\frac{1}{(2i\pi)^n} \int_{\mathbb{T}^n} \mathrm{Log}\,|P(x_1, \ldots, x_n)| \frac{dx_1}{x_1} \ldots \frac{dx_n}{x_n}\right),$$

where $\mathbb{T}^n = \{(z_1,\ldots,z_n) \in \mathbb{C}^n \mid |z_1| = \ldots = |z_n| = 1\}$ is the unit torus in dimension n. If $n = 1$, by Jensen's formula, it is given by (1.27). A function $f : \mathbb{R} \to \mathbb{R}$ is said quasiperiodic if it is the sum of finitely many periodic continuous functions. The function, defined for $k \geq 2$, $\mathrm{Li}_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}$, $|z| \leq 1$, is the kth-polylogarithm function [GlZ] [Lwn] [Za3]. For $x > 0$, $\mathrm{Log}^+ x$ denotes $\max\{0, \mathrm{Log}\, x\}$. Let $\mathscr{F}$ be an infinite subset of the set of nonzero algebraic numbers which are not a root of unity; we say that the *Conjecture of Lehmer is true for $\mathscr{F}$* if there exists a constant $c_{\mathscr{F}} > 0$ such that $\mathrm{M}(\alpha) \geq 1 + c_{\mathscr{F}}$ for all $\alpha \in \mathscr{F}$.

2 Lehmer's conjecture and Schinzel-Zassenhaus's conjecture: in different contexts

The problem of Lehmer raised up many works in number theory, and also in other domains where questions of minimality of analogues of the Mahler measure intervene. The questions related to totally real algebraic numbers, with minimality of the Weil height, are reported in § 7.3. The dynamics of algebraic numbers and its ergodic properties, related to the conjecture of Lehmer, are reported in § 4.3, after recalling the β-shift in § 4.1 and presenting the properties of Parry Upper functions in § 4.2.

2.1 Number theory: prime numbers, asymptotic expansions, minorations, limit points

The search for very large prime numbers has a long history. The method of linear recurrence sequences of numbers (Δ_m), typically satisfying

$$\Delta_{m+n+1} = A_1 \Delta_{m+1} + A_2 \Delta_{m+2} + \ldots + A_n \Delta_{m+n}, \tag{2.1}$$

in which prime numbers can be found, has been investigated from several viewpoints, by many authors [BuHV] [EtSTW] [EPSW]: in 1933 Lehmer [Le] developped an exhaustive approach from the Pierce numbers [Pi] $\Delta_n = \Delta_n(P) = \prod_{i=1}^{d}(\alpha_i^n - 1)$ of a monic integer polynomial P where α_i are the roots of P. The sequence (A_i) in (2.1) is then the coefficient vector of the integer monic polynomial which is the least common multiple of the $d+1$ polynomials: $P_{(0)}(x) = x - 1, P_{(1)}(x) = \prod_{i=1}^{d}(x - \alpha_i), P_{(2)}(x) = \prod_{i>j=1}^{d-1}(x - \alpha_i \alpha_j), \ldots, P_{(d)}(x) = x - \alpha_1 \alpha_2 \ldots \alpha_d$ (Theorem 13 in [Le]). Large prime numbers, possibly at a certain power, can be found in the factorizations of $|\Delta_n|$ that have large absolute values (Dubickas [Ds11], Ji and Qin [JiQn] in connection with Iwasawa theory). This can be done fairly quickly if the absolute values $|\Delta_n|$ do not increase too rapidly (slow growth rate). If P has no root on the unit circle, Lehmer proves

$$\lim_{n \to \infty} \frac{\Delta_{n+1}}{\Delta_n} = \mathrm{M}(P). \tag{2.2}$$

Einsiedler, Everest and Ward [ErEW] revisited and extended the results of Lehmer in terms of the dynamics of toral automorphisms ([EtW], Lind [Ld]). They considered expansive (no root on $|z| = 1$), ergodic (no α_i is a root of unity) and quasihyperbolic (if P is ergodic but not expansive) polynomials P and number theoretic heuristic arguments for estimating densities of primes in (Δ_n). In the quasihyperbolic case (for instance for irreducible Salem polynomials P), more general than the expansive case considered by Lehmer, (2.2) does not extend but the following more robust convergence law holds [Ld]:

$$\lim_{n\to\infty} \Delta_n^{1/n} = \mathrm{M}(P). \tag{2.3}$$

If P has a small Mahler measure, $< \Theta$, it is reciprocal by [Sy] and the quotients Δ_n/Δ_1 are perfect squares for all $n \geq 1$ odd. With $\Gamma_n(P) := \sqrt{\Delta_n/\Delta_1}$ in such cases, they obtain the existence of the limit

$$\lim_{j\to\infty} \frac{j}{\mathrm{Log}\,\mathrm{Log}\,\Gamma_{n_j}},$$

(n_j) being a sequence of integers for which Γ_{n_j} is prime, as a consequence of Merten's Theorem. This limit, say E_P, is likely to satisfy the inequality: $E_P \geq 2e^{\gamma}/\mathrm{Log}\,\mathrm{M}(P)$, where $\gamma = 0.577\ldots$ is the Euler constant. Moreover, by number-fields analogues of the heuritics for Mersenne numbers (Wagstaff, Caldwell), they suggest that the number of prime values of $\Gamma_{n_j}(P)$ with $n_j \leq x$ is approximately

$$\frac{2e^{\gamma}}{\mathrm{Log}\,\mathrm{M}(P)} \mathrm{Log}\,x.$$

This result shows the interest of having a polynomial P of small Mahler measure to obtain a sequence (Δ_n) associated with P very rich in primes. These authors consider many examples which fit coherently the heuristics. However, the discrepancy function is still obscure and reflects the deep arithmetics of the factorization of the integers $|\Delta_n|$ and of the quantities Γ_n. Let us recall the *problem of Lehmer*, as mentioned by Lehmer [Le] in 1933: *if ε is a positive quantity, to find a polynomial of the form*

$$f(x) = x^r + a_1 x^{r-1} + \ldots + a_r$$

where the a_is are integers, such that the absolute value of the product of those roots of f which lie outside the unit circle, lies between 1 *and* $1+\varepsilon$*... Whether or not the problem has a solution for* $\varepsilon < 0.176$ *we do not know.*

In general (§ 7.6 in [EPSW], [EtW2]), the Conjecture of Lehmer means that the growth rate of an integer linear recurrence sequence is uniformly bounded from below.

In view of understanding the size of the primes $p \geq 3$ found in (Δ_n) generated by the exhaustive method of the Pierce's numbers, Lehmer, in [Le2] (1977), established correlations between the Pierce's numbers $|\Delta_n|$ and the prime factors of the first factor of the class number of the cyclotomic fields $\mathbb{Q}(\xi_p)$ (ξ_p is a primitive pth root of

unity), using Kummer's formula, the prime factors being sorted out into arithmetic progressions: let $h(p)$ the class number of $\mathbb{Q}(\xi_p)$ and let $h^+(p)$ be the class number of of the real subfield $\mathbb{Q}(\xi_p + \xi_p^{-1})$. Kummer (1851) established that the ratio $h^-(p) = h(p)/h^+(p)$ is an integer, called *relative class number* or *first factor of the class number*, and that p divides $h(p)$ if and only if p divides $h^-(p)$. The factorization and the arithmetics of the large values of $h^-(p)$ is a deep problem [Ah] [FGleW] [Gle] [LeMy], related to class field theory in [My], where the validity of Kummer's conjectured asymptotic formula for $h^-(p)$ was reconsidered by Granville [Dee] [Gle].

The smallest Mahler measure $\mathrm{M}(\alpha)$ known, where α is a nonzero algebraic number which is not a root of unity, is Lehmer's number $= 1.17628\ldots$ (1.9), the smallest Salem number discovered by Lehmer [Le] in 1933 as dominant root of Lehmer's polynomial (1.10). Lehmer also discovered other small Salem numbers. Small Salem numbers were reinvestigated by Boyd in [Bo2] [Bo4] [Bo6], then by Flammang, Grandcolas and Rhin [FGR]. The search of small Mahler measures was reconsidered by Mossinghoff [Mf] [Mf2] then, using auxiliary functions, by Mossinghoff, Rhin and Wu [MRW]. For degrees up to 180, the list of Mossinghoff [Mlist] (2001), with contributions of Boyd, Flammang, Grandcolas, Lisonek, Poulet, Rhin and Sac-Epée [RnSE], Smyth, gives primitive, irreducible, noncyclotomic integer polynomials of degree at most 180 and of Mahler measure less than 1.3; this list is complete for degrees less than 40 [MRW], and, for Salem numbers, contains the list of the 47 known smallest Salem numbers, all of degree ≤ 44 [FGR].

Lehmer's conjecture is true (solved) in the following particular cases: (i) for the closed set S of Pisot numbers (Salem [Sa], Siegel [Si], Bertin et al [B-S]),

(ii) for the set of algebraic numbers α for which the minimal polynomial P_α is nonreciprocal by Smyth's Theorem [Sy] [Sy2] (1971) which asserts:

$$\mathrm{M}(\alpha) = \mathrm{M}(P_\alpha) \geq \Theta, \tag{2.4}$$

proved to be an isolated infimum by Smyth [Sy2],

(iii) for every nonzero algebraic integer $\alpha \in \mathbb{L}$, of degree d, assuming that $\mathbb{L}$ is a totally real algebraic number field, or a CM field (a totally complex quadratic extension of a totally real number field); then Schinzel [Sc2] obtained the minoration

$$\mathrm{M}(\alpha) \geq \Big(\frac{1+\sqrt{5}}{2}\Big)^{d/2}, \text{from which} : \mathrm{M}(\alpha) \geq ((1+\sqrt{5})/2)^{1/2} = 1.2720\ldots \tag{2.5}$$

Improvments of this lower bound, by Bertin, Rhin, Zaimi, are reported in § 7.3,

(iv) for α an algebraic number of degree d such that there exists a prime number $p \leq d \,\mathrm{Log}\, d$ that is not ramified in the field $\mathbb{Q}(\alpha)$; then Mignotte [Mt] [Mt2] showed: $\mathrm{M}(\alpha) \geq 1.2$; by extension, Silverman [Sn2] proved that the Conjecture of Lehmer is true if there exist primes $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_d$ in $\mathbb{Q}(\alpha)$ satisfying $N\mathfrak{p}_i \leq \sqrt{d \,\mathrm{Log}\, d}$,

(v) for any noncyclotomic irreducible polynomial P with all odd coefficients; Borwein, Dobrowolski and Mossinghoff [BDM] [GIMPW] proved (cf Theorem 2.2 and Silverman's Theorem 2.3 for details)

$$\mathrm{M}(P) \geq 5^{1/4} = 1.4953\ldots, \tag{2.6}$$

(vi) in terms of the Weil height, Amoroso and David [ADd2] proved that there exists a constant $c > 0$ such that, for all nonzero algebraic number α, of degree d, not being a root of unity, under the assumption that the extension $\mathbb{Q}(\alpha)/\mathbb{Q}$ is Galois, then

$$h(\alpha) \geq \frac{c}{d}. \tag{2.7}$$

Some minorations are known for some classes of polynomials (Panaitopol [Pol]). Bazylewicz [Bz] extended Smyth's Theorem (i.e. (2.4)) to polynomials over Kroneckerian fields K (i.e. for which $K/\mathbb{Q}$ is totally real or is a totally complex nonreal quadratic extension of such fields). Notari [Ni] and Lloyd-Smith [LSt] extended such results to algebraic numbers. Lehmer's problem is related to the minoration problem of the discriminant (Bertrand [Bed]). Mahler measures $\{\mathrm{M}(\alpha) \mid \alpha \in \overline{\mathbb{Q}}\}$ are Perron numbers, by the work of Adler and Marcus [AM] in topological dynamics, as a consequence of the Perron-Frobenius theory. The set $\mathbb{P}$ of Perron numbers is everywhere dense in $[1, +\infty)$ and is important since it contains subcollections which have particular topological properties for which conjectures exist. The set $\mathbb{P}$ admits a nonfactorial multiplicative arithmetics [Bte] [Ld2] [VG3] for which the restriction of the usual addition $+$ to a given subcollection is not necessarily internal [Ds10]. Salem [Sa2] proved that $S \subset \mathbb{P}$ is closed, and that $S \subset \overline{T}$. Boyd [Bo] conjectured that $T \cup S$ is closed and that $S = (S \cup T)^{(1)}$ ([Bo2], p. 237). This second Conjecture would imply that all Salem numbers $< \Theta$ would also be isolated Salem numbers, not only Lehmer's number. The set of Mahler measures $\{\mathrm{M}(\alpha) \mid \alpha \in \overline{\mathbb{Q}}\}$ and the semi-group $\{\mathrm{M}(P) \mid P(X) \in \mathbb{Z}[X]\}$ are strict subsets of $\mathbb{P}$ and are distinct (Boyd [Bo10], Dubickas [Ds7] [Ds8]). Values of Mahler measures were studied by Boyd [Bo9] [Bo10] [Bo11], Chern and Vaaler [CV], Dixon and Dubickas [DDs], Dubickas [Ds6], Schinzel [Sc4], Sinclair [Si]. Boyd [Bo11] has shown that the Perron numbers γ_n which are the dominant roots of the height one (irreducible) trinomials $-1-z+z^n$, $n \geq 4$, are not Mahler measures. The *inverse problem* for the Mahler measure consists in determining whether, or not, a Perron number γ is the Mahler measure $\mathrm{M}(P)$ of an integer polynomial P, and to give formulas for the number $\#\{P \mid \mathrm{M}(P) = \gamma\}$ of such polynomials with measure γ and given degree (Boyd [Bo9], Dixon and Dubickas [DDs], Staines [Sts]). Drungilas and Dubickas [DrDs] and Dubickas [Ds6] [Ds8] proved that the subset of Mahler measures is very rich: namely, for any Perron number β, there exists an integer $n \in \mathbb{N}$ such that $n\beta$ is a Mahler measure, and any real algebraic integer is the difference of two Mahler measures.

The set of limit points of $\{\mathrm{M}(P) \mid P(X) \in \mathbb{Z}[X]\}$ is obtained by the following useful Theorem of Boyd and Lawton [Bo7] [Bo8] [Lw], which correlates Mahler measures of univariate polynomials to Mahler measures of multivariate polynomials:

Theorem 2.1. *let* $P(x_1, x_2, \ldots, x_n) \in \mathbb{C}[x_1, \ldots, x_n])$ *and* $\underline{r} = (r_1, r_2, \ldots, r_n)$, $r_i \in \mathbb{N}_{>0}$. *Let* $P_{\underline{r}}(x) := P(x^{r_1}, x^{r_2}, \ldots, x^{r_n})$. *Let*

$$q(\underline{r}) := \min\{H(\underline{t}) \mid \underline{t} = (t_1, t_2, \ldots, t_n) \in \mathbb{Z}^n, \underline{t} \neq (0, \ldots 0), \ \sum_{j=1}^{n} t_j r_j = 0\},$$

where $H(\underline{t}) = \max\{|t_i| \mid 1 \leq j \leq n\}$. *Then*

$$\lim_{q(\underline{r})\to\infty} \mathrm{M}(P_{\underline{r}}) = \mathrm{M}(P).$$

This theorem allows the search of small limit points of (univariate) Mahler's measures, by several methods [BM]; another class of methods relies upon the (EM) Expectation-Maximization algorithm [MlK] [EoRSe]. The set of limit points of the Salem numbers was investigated either by the "Construction of Salem" [Bo2] [Bo3] or sets of solutions of some equations [BPa]. Everest [Et], then Condon [Cdn] [Cdn2] established asymptotic expansions of the ratio $\mathrm{M}(P_{\underline{r}})/\mathrm{M}(P)$. For bivariate polynomials $P(x,y) \in \mathbb{C}[x,y]$ such that P and $\partial P/\partial y$ do not have a common zero on $\mathbb{T}\times\mathbb{C}$, then Condon (Theorem 1 in [Cdn2]) establishes the expansion, for k large enough,

$$\mathrm{Log}\left(\frac{\mathrm{M}(P_{\underline{r}})}{\mathrm{M}(P)}\right) = \mathrm{Log}\left(\frac{\mathrm{M}(P(x,x^n))}{\mathrm{M}(P(x,y))}\right) = \sum_{j=2}^{k-1} \frac{c_j}{n^j} + O_{P,k}\left(\frac{1}{n^k}\right), \tag{2.8}$$

(n is not the degree of the univariate polynomial $P(x,x^n)$) where the coefficients c_j are values of a quasiperiodic function of n, as finite sums of real and imaginary parts of values of Li_a polylogarithms, $2 \leq a \leq j$, weighted by some rational functions deduced from the derivatives of P, where the sums are taken over algebraic numbers deduced from the intersection of $\mathbb{T}^2$ and the hypersurface of $\mathbb{C}^2$ defined by P (affine zero locus). In particular, if P is an integer polynomial, the coefficients c_j are $\overline{\mathbb{Q}}$-linear combinations of polylogarithms evaluated at algebraic arguments. For instance, for $P(x,y) = -1+x+y$, $G_n(x) = -1+x+x^n$, the coefficient $c_2(n)$ in the expansion of $\mathrm{Log}(\mathrm{M}(G_n)/\mathrm{M}(P))$, though a priori quasiperiodic, is a periodic function of n modulo 6 which can be directly computed (Theorem 1.3 in [VG6]), as: for n odd:

$$c_2(n) = \begin{cases} \sqrt{3}\pi/18 = +0.3023\ldots & \text{if } n \equiv 1 \text{ or } 3 \pmod{6}, \\ -\sqrt{3}\pi/6 = -0.9069\ldots & \text{if } n \equiv 5 \pmod{6}, \end{cases}$$

for n even:

$$c_2(n) = \begin{cases} -\sqrt{3}\pi/36 = -0.1511\ldots & \text{if } n \equiv 0 \text{ or } 4 \pmod{6}, \\ +\sqrt{3}\pi/12 = +0.4534\ldots & \text{if } n \equiv 2 \pmod{6}. \end{cases}$$

For the height one trinomial $1+x+y$, Corollary 2 in [Cdn2] gives the coefficients c_j, $j \geq 2$, as linear combinations of polylogaritms evaluated at third roots of unity, with coefficients coming from the Stirling numbers of the first and second kind, i.e. in $\frac{1}{2\pi}\mathbb{Z}[\sqrt{3}]$. The method of Condon also provides the other coefficients c_j, $j \geq 3$, for the trinomial $-1+x+y$ in the same way.

Doche in [Dhe] obtains an alternate method to Boyd-Lawton's Theorem, in the objective of obtaining estimates of the Mahler measures of bivariate polynomials: let $P(y,z) \in \mathbb{C}[y,z]$ be a polynomial such that $\deg_z(P) > 0$, let $\xi_n := e^{\frac{2i\pi}{n}}$ and assume that

$P(\xi_n^k, z) \not\equiv 0$ for all n, k. Then

$$\mathrm{M}(P(y,z))^{1/\deg_z(P)} = \lim_{n\to\infty} \mathscr{M}\Big(\prod_{k=1}^{n} P(\xi_n^k, z)\Big) \tag{2.9}$$

(n is not the degree of the univariate polynomial $\prod_{k=1}^{n} P(\xi_n^k, x)$). Doche's and Condon's methods cannot be used for the problem of Lehmer since 1 does not belong to the first derived set of the set of Mahler measures of univariate integer polynomials (assuming true Lehmer's Conjecture).

Algebraic numbers close to 1 ask many questions [Do3] and require new methods of investigation, as reported by Amoroso [A2]. For α an algebraic integer of degree $d > 1$, not a root of unity, Blansky and Montgomery [ByM] showed, with multivariate Fourier series,

$$\mathrm{M}(\alpha) > 1 + \frac{1}{52}\frac{1}{d\,\mathrm{Log}\,(6d)}.$$

By a different approach, using an auxiliary function and a proof of transcendence (Thue's method), Stewart [St] obtained the same minoration but with a constant $c \neq 1/52$ instead of $1/52$ [BgMeNn] [La2] [MtW] [W]. In 1979 a remarkable minoration was obtained by Dobrowolski [Do2] who showed

$$\mathrm{M}(\alpha) \;>\; 1 + (1-\varepsilon)\left(\frac{\mathrm{Log}\,\mathrm{Log}\,d}{\mathrm{Log}\,d}\right)^3, \quad d > d_1(\varepsilon). \tag{2.10}$$

for any nonzero algebraic number α of degree d, with $1-\varepsilon$ replaced by $1/1200$ for an effective version (then with $d \geq 2$), in particular for $|\alpha| > 1$ arbitrarily close to 1. The minoration (2.10) was also obtained by Mignotte in [Mt] [Mt2] but with a constant smaller than $1-\varepsilon$. Cantor and Strauss [CS] [Pxe], then Louboutin [Lt], improved the constant $1-\varepsilon$: they obtained $2(1+o(1))$, resp. $9/4$ (cf also Rausch [Ra] and Lloyd-Smith [LSt]). If α is a nonzero algebraic number of degree $d \geq 2$, Voutier [V] obtained the better effective minorations:

$$\mathrm{M}(\alpha) > 1 + \frac{1}{4}\left(\frac{\mathrm{Log}\,\mathrm{Log}\,d}{\mathrm{Log}\,d}\right)^3 \qquad \text{and} \qquad \mathrm{M}(\alpha) > 1 + \frac{2}{(\mathrm{Log}\,(3d))^3}. \tag{2.11}$$

For sufficiently large degree d, Waldschmidt ([W2], Theorem 3.17) showed that the constant $1-\varepsilon$ could be replaced in (2.10) by $1/250$ with a transcendence proof which uses an interpolation determinant. It is remarkable that these minorations only depend upon the degree of α and not of the size of the coefficients, i.e. of the (naïve) height of their minimal polynomial. Dobrowolski's proof is a transcendence proof (using Siegel's lemma, extrapolation at finite places) which has been extended to the various Lehmer problems (§ 2.2).

In 1965 the following Conjecture was formulated in [SZ].

Conjecture 4 (Schinzel Zassenhaus). *Denote by* $\mathrm{m}_h(n)$ *the minimum of the houses* $\lceil\alpha\rceil$ *of the algebraic integers* α *of degree n which are not a root of unity. There exists*

a (universal) constant $C > 0$ *such that*

$$\mathrm{m}_h(n) \geq 1 + \frac{C}{n}, \qquad n \geq 2. \tag{2.12}$$

An algebraic integer α, of degree n, is said *extremal* if $\lceil\alpha\rceil = \mathrm{m}_h(n)$. An extremal algebraic integer is not necessarily a Perron number [BMs2].

Schinzel and Zassenhaus [SZ] obtained the first result: for $\alpha \neq 0$ being an algebraic integer of degree $n \geq 2$ which is not a root of unity, then $\lceil\alpha\rceil > 1 + 4^{-(s+2)}$, where $2s$ is the number of nonreal conjugates of α. For a nonreciprocal algebraic integer α of degree n, Cassels [Cs] obtained:

$$\lceil\alpha\rceil > 1 + \frac{c_2}{n}, \qquad \text{with } c_2 = 0.1; \tag{2.13}$$

Breusch [Br] independently showed that $c_2 = \mathrm{Log}\,(1.179\ldots) = 0.165\ldots$ could be taken; Schinzel [Sc2] showed that $c_2 = 0.2$ could also be taken. Finally Smyth [Sy] improved the minoration (2.13) with $c_2 = \mathrm{Log}\,\Theta = 0.2811\ldots$. On the other hand, Boyd [Bo9] showed that c_2 cannot exceed $\frac{3}{2}\mathrm{Log}\,\Theta = 0.4217\ldots$. In 1997 Dubickas [Ds3] showed that $c_2 = \omega - \varepsilon$ with $\omega = 0.3096\ldots$ the smallest root of an equation in the interval $(\mathrm{Log}\,\Theta, +\infty)$, with $\varepsilon > 0$, $n_0(\varepsilon)$ an effective constant, and for all $n > n_0(\varepsilon)$. These two bounds seem to be the best known extremities delimiting the domain of existence of the constant c_2 [DoLS].

The expression of the minorant in (2.12), "in $1/n$", as a function of n, is not far from being optimal, being "in $1/n^2$" at worse in (2.14). Indeed, for nonzero algebraic integers α, Kronecker's Theorem [Krr] implies that $\lceil\alpha\rceil = 1$ if and only if α is a root of unity. The sufficient condition in Kronecker's Theorem was weakened by Blansky and Montgomery [ByM] who showed that α, with $\deg\alpha = n$, is a root of unity provided

$$\lceil\alpha\rceil \leq 1 + \frac{1}{30n^2\,\mathrm{Log}\,(6n)}.$$

Dobrowolsky [Do] sharpened this condition by: if

$$\lceil\alpha\rceil < 1 + \frac{\mathrm{Log}\,n}{6n^2}, \tag{2.14}$$

then α is a root of unity. Matveev [Mv] proved, for α, with $\deg\alpha = n$, not being a root of unity,

$$\lceil\alpha\rceil \geq \exp\frac{\mathrm{Log}\,(n + \frac{1}{2})}{n^2}.$$

Rhin and Wu [RnW] verified Schinzel Zassenhaus's Conjecture up to $n = 28$ and improved Matveev's minoration as:

$$\lceil\alpha\rceil \geq \exp\frac{3\,\mathrm{Log}\,(\frac{n}{3})}{n^2} \qquad 4 \leq n \leq 12,$$

and, for $n \geq 13$,

$$\lceil\alpha\rceil \geq \exp \frac{3\mathrm{Log}\,(\frac{n}{2})}{n^2}.$$

Matveev's minoration is better than Voutier's lower bound [V]

$$m_h(n) \geq \left(1 + \frac{1}{4}\left(\frac{\mathrm{Log}\,\mathrm{Log}\, n}{\mathrm{Log}\, n}\right)^3\right)^{1/n}$$

for $n \leq 1434$, and Rhin Wu's minoration is better than Voutier's bound for $13 \leq n \leq 6380$. For reciprocal nonzero algebraic integers α, $\deg(\alpha) = n \geq 2$, not being a root of unity, Dobrowolski's lower bound is

$$\lceil\alpha\rceil > 1 + (2-\varepsilon)\left(\frac{\mathrm{Log}\,\mathrm{Log}\, n}{\mathrm{Log}\, n}\right)^3 \frac{1}{n}, \qquad n \geq n_0(\varepsilon),$$

where the constant $2-\varepsilon$ could be replaced by $\frac{9}{2}-\varepsilon$ (Louboutin [Lt]), or, better, by $\frac{64}{\pi^2}-\varepsilon$ (Dubickas [Ds]). Callahan, Newman and Sheingorn [CNSn] introduce a weaker version of Schinzel Zassenhaus's Conjecture: given a number field K, they define the *Kronecker constant* of K as the least $\eta_K > 0$ such that $\lceil\alpha\rceil \geq 1 + \eta_K$ for all $\alpha \in K$. Under certain assumptions on K, they prove that there exists $c > 0$ such that $\eta_K \geq c/[K:\mathbb{Q}]$.

The sets of extremal algebraic integers are still unknown. In Boyd [Bo7] [Wu] the following conjectures on *extremality* are formulated:

Conjecture 5 (Lind - Boyd). *The smallest Perron number of degree $n \geq 2$ has minimal polynomial*

$$\begin{array}{ll} X^n - X - 1 & \text{if } n \not\equiv 3,5 \mod 6, \\ (X^{n+2} - X^4 - 1)/(X^2 - X + 1) & \text{if } n \equiv 3 \mod 6, \\ (X^{n+2} - X^2 - 1)/(X^2 - X + 1) & \text{if } n \equiv 5 \mod 6. \end{array}$$

Conjecture 6 (Boyd). *(i) If α is extremal, then it is always nonreciprocal,*

(ii) if $n = 3k$, then the extremal α has minimal polynomial

$$X^{3k} + X^{2k} - 1, \qquad \text{or} \quad X^{3k} - X^{2k} - 1,$$

(iii) the extremal α of degree n has asymptotically a number of conjugates $\alpha^{(i)}$ outside the closed unit disc equal to

$$\cong \frac{2}{3}n, \qquad n \to \infty.$$

This asymptotic proportion of $\frac{2}{3}n$ would correspond to a fairly regular angular distribution of the complete set of conjugates in a small annulus containing the unit circle, in the sense of the Bilu-Erdős-Turán -Amoroso-Mignotte equidistribution theory [AM] [Bebu] [Bu] [ET]. As for the trinomials (G_n) Conjecture 6 (iii) is compatible with the fact that the Perron numbers $\theta_n^{-1}, n \neq 2,3$, are not extremal since the

prorata of roots of G_n^* of modulus > 1 is asymptotically $1/3$ and not $2/3$ (Proposition 5.1 in [VG6]).

The nature of the coefficient vector of an integer polynomial P is linked to the Mahler measure $\mathrm{M}(P)$ and to extremal properties [Mhr]. If some inequalities between coefficients occur, then Brauer [Brr] proved that P is a Pisot polynomial; in this case Lehmer's problem is solved for this class $\{P\}$. Stankov [Stv] proved that a real algebraic integer $\tau > 1$ is a Salem number if and only if its minimal polynomial is reciprocal of even degree ≥ 4 and if there is an integer $n \geq 2$ such that τ^n has minimal polynomial $P_n(x) = a_{0,n} + a_{1,n}x + \ldots + a_{d,n}x^n$ which is also reciprocal of degree d and satisfies the condition

$$|a_{d-1,n}| > \frac{1}{2}\big(\frac{d}{d-2}\big)(2 + \sum_{k=2}^{d-2} |a_{k,n}|).$$

Related to Kronecker's Theorem [Krr] is the problem of finding necessary and sufficient conditions on the coefficient vector of reciprocal, self-inversive, resp. self-reciprocal polynomials to have all their roots on the unit circle (unimodularity): Lakatos [Los3] proved that a polynomial $P(x) = \sum_{j=0}^{m} A_j x^j \in \mathbb{R}[x]$ satisfying the conditions $A_{m-j} = A_j$ for $j \leq m$ and

$$|A_m| \geq \sum_{j=0}^{m} |A_j - A_m|$$

has all zeroes on the unit circle. Schinzel [Sc5], Kim and Park [KmPk], Kim and Lee [KmLe], Lalin and Smyth [LinSy] obtained generalizations of this result. Suzuki [Szi] established correlations between this problem and the theory of canonical systems of ordinary linear differential equations. Lakatos and Losonczi [LosLi] [LosLi2] proved that, for a self-inversive polynomial $P_m(z) = \sum_{j=0}^{m} A_k z^k \in \mathbb{C}[z]$, $m \geq 1$, the roots of P_m are all on the unit circle if $|A_m| \geq \sum_{k=1}^{m-1} |A_k|$; moreover if this inequality is strict then the zeroes $e^{i\varphi_l}$, $l = 1, \ldots, m$, are simple and can be arranged such that, with $\beta_m = \arg(A_m \left(\overline{A_0}/A_m\right)^{1/2})$,

$$\frac{2((l-1)\pi - \beta_m)}{m} < \varphi_l < \frac{2(l\pi - \beta_m)}{m}.$$

In the direction of Salem polynomials, ν-Salem polynomials and more [Kda] [Set], a generalization was obtained by Vieira [Vra]: if a sufficient condition is satisfied then a self-inversive polynomial has a fixed number of roots on the unit circle. Namely, let $P(z) = a_n z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0 \in \mathbb{C}[z]$, $a_n \neq 0$, be such that $P(z) = \omega\, z^n\, \overline{P}(1/z)$ with $|\omega| = 1$. If the inequality

$$|a_{n-l}| > \frac{1}{2}\left(\frac{n}{n-2l}\right) \sum_{k=0, k\neq l, k\neq n-l}^{n} |a_k|, \qquad l < n/2$$

is satisfied, then $P(z)$ has exactly $n-2l$ roots on the unit circle and these roots are simple; moreover, if n is even and $l = n/2$, then $P(z)$ has no root on $|z| = 1$ if the inequality $|a_{n/2}| > \sum_{k=0,k\neq n/2}^{n} |a_k|$ is satisfied.

Questions of irreducibility of P as a function of the coefficient vector were studied in [Ds5]. Flammang [Fg2] obtained new inequalities for the Mahler measure $\mathrm{M}(P)$ [W3], and Flammang, Rhin and Sac-Epée [FRSC] proved relations between the integer transfinite diameter and polynomials having a small Mahler measure. The lacunarity of P and the minoration of $\mathrm{M}(P)$ are correlated: when P is a noncyclotomic (sparse) integer polynomial, Dobrowolski, Lawton and Schinzel [DoLS], then Dobrowolski [Do4] [Do5], obtained lower bounds of $\mathrm{M}(P)$ as a function of the number k of its nonzero coefficients: e.g. in [Do5], with $a < 0.785$,

$$\mathrm{M}(P) \geq 1 + \frac{1}{\exp(a 3^{\lfloor (k-2)/4 \rfloor} k^2 \operatorname{Log} k)},$$

and, if P is irreducible,

$$\mathrm{M}(P) \geq 1 + \frac{0.17}{2^{\lfloor k/2 \rfloor} \lfloor k/2 \rfloor !}.$$

Dobrowolski, then McKee and Smyth [MS2] obtained minorants of $\mathrm{M}(P)$ for the reciprocal polynomials $P(z) = z^n D_A(z + 1/z)$ where D_A is the characteristic polynomial of an integer symmetric $n \times n$ matrix A; McKee and Smyth obtained $\mathrm{M}(P) = 1$ or $\mathrm{M}(P) \geq 1.176280\ldots$ (Lehmer's number) solving the problem of Lehmer for the family of such polynomials. Dobrowolski (2008) proved that many totally real integer polynomials P cannot be represented by integer symmetric matrices A, disproving a conjecture of Estes and Guralnick. Dubickas and Konyagin [Ds13] [DsKn] studied the number of integer polynomials as a function of their (naïve) height and resp. their Mahler measure. The next two theorems show that Lehmer's Conjecture is true for the set of the algebraic integers which are the roots of polynomials in particular families of monic integer polynomials.

Theorem 2.2 (Borwein, Dobrowolski, Mossinghoff [BDM]). *Let $m \geq 2$, and let $f(X) \in \mathbb{Z}[X]$ be a monic polynomial of degree D with no cyclotomic factors that satisfies*

$$f(X) \equiv X^D + X^{D-1} + \ldots + X^2 + X + 1 \qquad \bmod m.$$

Then

$$\sum_{f(\alpha)=0} h(\alpha) \geq \frac{D}{D+1} C_m,$$

where we may take

$$C_2 = \frac{1}{4} \operatorname{Log} 5 \qquad \textit{and} \quad C_m = \operatorname{Log} \frac{\sqrt{m^2+1}}{2} \quad \textit{for } m \geq 3.$$

Theorem 2.3 (Silverman [Sn5]). *For all $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ with the following property: let $f(X) \in \mathbb{Z}[X]$ be a monic polynomial of degree D such that*

$$f(X) \text{ is divisible by } X^{n-1} + X^{n-2} + \ldots + X + 1 \quad \text{in } (\mathbb{Z}/m\mathbb{Z})[X].$$

for some integers $m \geq 2$ and $n \geq \max\{\varepsilon D, 2\}$. Suppose further that no root of $f(X)$ is a root of unity. Then

$$\sum_{f(\alpha)=0} h(\alpha) \geq C_\varepsilon \operatorname{Log} m.$$

Limit points of Mahler measures of univariate polynomials are algebraic numbers or transcendental numbers: by (2.9) and Theorem 2.1, they are Mahler measures of multivariate polynomials. The problem of finding a positive lower bound of the set of such limit points of Mahler measures is intimately correlated to the problem of Lehmer [Sc]. Smyth (1971)[Sy4] found the remarkable identity: $\operatorname{Log} \mathrm{M}(1 + x + y) = \Lambda$ (given by (1.16)). The values of logarithmic Mahler measures of multivariate polynomials are sums of special values of different L-functions, often conjecturally [Bo16]; the remarkable conjectural identities discovered by Boyd in [Bo16] (1998), also by Smyth [Sy4] and Ray [Ry], serve as starting points for further studies, some of them being now proved, e.g. [Lin] [Lin2] [Rgs] [ShVo] [Zun].

Indeed, after the publication of [Bo16], Deninger [Dgr] reinterpreted the logarithmic Mahler measures $\operatorname{Log} \mathrm{M}(P)$ of Laurent polynomials $P \in \mathbb{Z}[\mathbb{Z}^n] = \mathbb{Z}[x_1^{\pm}, \ldots, x_n^{\pm}]$ as topological entropies in the theory of dynamical systems of algebraic origin, with $\mathbb{Z}^n$-actions (Schmidt [Sdt2], Chap. V, Theorem 18.1; Lind, Schmidt and Ward [LdSW]). This new approach makes a link with higher K-theory, mixed motives (Deninger [Dgr2]), real Deligne cohomology, the Bloch-Beilinson conjectures on special values of L- functions, and Mahler measures. There are two cases: either P does not vanish on $\mathbb{T}^n$, in which case $\operatorname{Log} \mathrm{M}(P)$ is a Deligne period of the mixed motive over $\mathbb{Q}$ which corresponds to the nonzero symbol $\{P, x_1, \ldots, x_n\}$ (Theorem 2.2 in [Dgr]), or, if P vanishes on $\mathbb{T}^n$, under some assumptions, it is a difference of two Deligne periods of certain mixed motives, equivalently, the difference of two symbols evaluated against topological cycles ("integral K-theory cycles") (Theorem 3.4 in [Dgr], with a motivic reinterpretation in Theorem 4.1 in [Dgr]).

Let $\mathbb{G}_{m,A}^n := \operatorname{Spec}(A[\mathbb{Z}^n])$ be the split n-torus defined over the commutative ring $A = \mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$. The polynomial $P \not\equiv 0$ defines the irreducible closed subscheme $Z := \operatorname{Spec}(\mathbb{Z}[\mathbb{Z}^n]/(P)) \subset \mathbb{G}_{m,\mathbb{Z}}^n, Z \neq \mathbb{G}_{m,\mathbb{Z}}^n$, For any coherent sheaf $\mathscr{F}$ on $\mathbb{G}_{m,A}^n$, the group $\Gamma(\mathbb{G}_{m,A}^n, \mathscr{F})$ of global sections, equipped with the discrete topology, admits a Pontryagin dual $\Gamma(\mathbb{G}_{m,A}^n, \mathscr{F})^*$ which is a compact group. This compact group endowed with the canonical $\mathbb{Z}^n$-action constitute an arithmetic dynamical system for which the entropy can be defined according to [Sdt2], and correlated to the Mahler measure (Theorem 18.1 in [Sdt2]); the application to P, $A = \mathbb{Z}$ and $\mathscr{F} = \mathscr{O}_Z$ provides the identity with the entropy: $h(\mathscr{O}_Z) = \operatorname{Log} \mathrm{M}(P)$. The definition

$$\operatorname{Log} \mathrm{M}(P) := \frac{1}{(2i\pi)^n} \int_{\mathbb{T}^n} \operatorname{Log} |P(x_1, \ldots, x_n)| \frac{dx_1}{x_1} \ldots \frac{dx_n}{x_n}$$

corresponds to the integration of a differential form in connection with the cup-product $\mathrm{Log}\,|P| \cup \mathrm{Log}\,|x_1| \cup \ldots \cup \mathrm{Log}\,|x_n|$ in the real Deligne cohomology of $\mathbb{G}^n_{m,\mathbb{R}} \setminus Z_{\mathbb{R}}$. The link between the L-series $L(M,s)$ of a motive M, and its derivatives, Deligne periods, and the Beilinson conjectures, comes from the Conjecture of Deligne-Scholl ([Dgr] Conjecture 2.1). Further, Rodriguez-Villegas [RVs] studied the conditions of applicability of the conjectures of Bloch-Beilinson for having logarithmic Mahler measures $\mathrm{Log}\,\mathrm{M}(P)$ expressed as L-series.

The example of $\mathrm{Log}\,\mathrm{M}(P(x_1,x_2)) = \mathrm{Log}\,\mathrm{M}((x_1+x_2)^2+k)$, with $k \in \mathbb{N}$, is computed in Proposition 19.10 in [Sdt2]. For instance, for $k=3$, we have

$$\mathrm{Log}\,\mathrm{M}((x_1+x_2)^2+3) = \frac{2}{3}\mathrm{Log}\,3 + \frac{\sqrt{3}}{\pi} L(2,\chi_3).$$

Deninger shows (ex. [Dgr] p. 275) the cohomological origin of each term: $\frac{\sqrt{3}}{\pi}L(2,\chi_3)$ from the first $\mathscr{M}$-cohomology group $H^1_{\mathscr{M}}(\partial A, \mathbb{Q}(2))$, $\frac{2}{3}\mathrm{Log}\,3$ from the second $\mathscr{M}$-cohomology group $H^2_{\mathscr{M}}(Z^{\mathrm{reg}}, \mathbb{Q}(2))$. Bornhorn [Brn], and later Standfest [Sst], reinvestigated further the conjectural identities of Boyd [Bo16] in particular the formulas of mixed type, containing several types of L-series. The logarithmic Mahler measure $\mathrm{Log}\,\mathrm{M}(P)$ is then written $= *L'(s_1,\chi) + *L'(E,s_2)$, where χ is a Dirichlet character, $L(s_1,\chi)$ the corresponding Dirichlet series, $L(E,s_2)$ the Hasse-Weil L-function of an elliptic curve $E/\mathbb{Q}$ deduced from P, and s_1, s_2 algebraic numbers. Following Deninger and Rodriguez-Villegas, Lalin [Lin] [Lin2] introduces techniques for applying Goncharov's constructions of the regulator on polylogarithmic motivic complexes in the objective of computing Mahler measures of multivariate Laurent polynomials. With some three-variable polynomials, whose zero loci define singular $K3$ surfaces, Bertin et al [B-Ms] prove that the logarithmic Mahler measure is of the form $*L'(g,O) + *L'(\chi,-1)$ where g is the weight 4 newform associated with the $K3$ surface and χ is a quadratic character. Other three-variable Mahler measures are associated with special values of modular and Dirichlet L-series [Srt]. Some four-variables polynomials define a Calabi-Yau threefold and the logarithmic Mahler measure is of the form $*L'(f,O) + *\zeta'(-2)$ where f is a normalized newform deduced from the Dedekind eta function [PRS]. Multivariable Mahler measures are also related to mirror symmetry and Picard - Fuchs equations in Zhou [Zou].

In comparison, the limit points of the set S of Pisot numbers were studied by analytical methods by Amara [Ama]. The set of values $\{\mathrm{Log}\,\mathrm{M}(P) \mid P \in \mathbb{Z}[\mathbb{Z}^n], n \geq 1\}$ is conjecturally (Boyd [Bo8]) a closed subset of $\mathbb{R}$ for the usual topology.

2.2 Small points and Lehmer problems in higher dimension

The theory of heights [BriG] [Scl] [W3] is a powerful tool for studying distributions of algebraic numbers, algebraic points on algebraic varieties, and of subvarieties in projective spaces by extension. Points having a small height, or "small points", resp.

"small" projective varieties, together with their distribution, have a particular interest in the problem of Lehmer in higher dimension.

In the classical Lehmer problem, the "height" is the Weil height, and Lehmer's Conjecture is expressed by a Lehmer inequality where the minorant is "a function of the degree", i.e. it states that there exists a universal constant $c > 0$ such that

$$h(\alpha) \geq \frac{c}{\deg(\alpha)} \tag{2.15}$$

unless $\alpha = 0$ or is a root of unity. The generalizations of Lehmer's problem are still formulated by a minoration as in (2.15), but in which "α" is replaced by a rational point "P" of a (abelian) variety, or replaced by a variety "V", where "h" is replaced by another height, more suitable, where the degree "$\deg(\alpha)$" may be replaced by the more convenient "*obstruction index*" ("degree of a variety"), where the minorant function of the "degree" may be more sophisticated than the inverse "$\deg(\alpha)^{-1}$". These different minoration forms extend the classical Lehmer's inequality into a Lehmer type inequality. Generalizing Lehmer's problem separates into three different Lehmer problems:

(i) the classical Lehmer problem,

(ii) the relative Lehmer problem,

(ii) Lehmer's problem for subvarieties.

(i) *The classical Lehmer problem*: on $\mathbb{G}_m$, Dobrowolski's and Voutier's minorations, given by (1.14) and (2.11), with "$(\mathrm{Log} \deg(\alpha))^3$" at the denominator, were up till now considered as the best general lower bounds, as functions of the degree $\deg(\alpha)$. Generalizations to higher dimension (below) have been largely studied: e.g. Amoroso and David [ADd2] [ADd4] [ADd5], Pontreau [Pru] [Pru2], W. Schmidt [Swt3] for points on $\mathbb{G}_m^n$, Anderson and Masser [AnMr], David [Did2], Galateau and Mahé [GM], Hindry and Silverman [HyS], Laurent [La], Silverman [Sn] [Sn2] [Sn4] for elliptic curves, David and Hindry [Did] [DH], Masser [Mr4] for abelian varieties.

Conjecture 7. *(Elliptic Lehmer problem) Let E/K be an elliptic curve over a number field K. There is a positive constant $C(E) > 0$ such that, if $P \in E(\overline{K})$ has infinite order,*

$$\widehat{h}(P) \geq \frac{c(E)}{[K(P):K]}. \tag{2.16}$$

Theorem 2.4 (Laurent [La]). *Let E/K be an elliptic curve with complex multiplication over a number field K. There is a positive constant $c(E/K)$ such that*

$$\widehat{h}(P) \geq \frac{c(E/K)}{D} \left(\frac{\mathrm{Log}\,\mathrm{Log}\, 3D}{\mathrm{Log}\, 2D} \right)^3 \qquad \textit{for all } P \in E(\overline{K}) \setminus E_{\mathrm{tors}} \tag{2.17}$$

where $D = [K(P):K]$.

Masser [Mr] [Mr2] [Mr4], and David [Did2], gave estimates of lower bounds of $\widehat{h}(P)$ for elliptic curves and abelian varieties, on families of abelian varieties [Mr3],

for P of infinite order. Galateau and Mahé [GM] solved the elliptic Lehmer problem in the Galois case, extending Amoroso David's Theorem ([ADd2], and [AVa2] for sharper estimates):

Theorem 2.5 (Galateau - Mahé [GM]). *Let E/K be an elliptic curve over a number field K. There is a positive constant $C(E) > 0$ such that, if $P \in E(\overline{K})$ has infinite order and the field extension $K(P)/K$ is Galois,*

$$\widehat{h}(P) \geq \frac{c(E)}{[K(P):K]}. \tag{2.18}$$

Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{G}_m^n(\overline{\mathbb{Q}}) \subset \mathbb{P}^n(\overline{\mathbb{Q}})$. The height of α in $\mathbb{G}_m^n(\overline{\mathbb{Q}})$ is defined by $h(\alpha) = h(1 : \alpha)$ the absolute logarithmic height. Let $F_0 \in \mathbb{Q}[x_1, \dots, x_n]$ be a nonzero polynomial vanishing at α. The *obstruction index* of α is by definition $\deg(F_0)$, denoted by $\delta_{\mathbb{Q}}(\alpha)$.

Conjecture 8. *(Multiplicative Lehmer problem) For any integer $n \geq 1$, there exists a real number $c(n) > 0$ such that*

$$h(\alpha) \geq \frac{c(n)}{\delta_{\mathbb{Q}}(\alpha)} \tag{2.19}$$

for all $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{G}_m^n(\overline{\mathbb{Q}})$ such that $\alpha_1, \dots, \alpha_n$ are multiplicatively independent.

Small points of subvarieties of algebraic tori were studied by Amoroso [A4].

Theorem 2.6 (Amoroso - David). *There exist a positive constant $c(n) > 0$ such that, for all $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{G}_m^n(\overline{\mathbb{Q}})$ such that $\alpha_1, \dots, \alpha_n$ are multiplicatively independent,*

$$h(\alpha) \geq \frac{c(n)}{\delta_{\mathbb{Q}}(\alpha)} \left(\mathrm{Log}\,(3\delta_{\mathbb{Q}}(\alpha))\right)^{-\eta(n)} \tag{2.20}$$

with $\eta(n) = (n+1)(n+1)!^n - n$.

As a consequence of the main Theorem in [AVa2] Amoroso and Viada improved the preceding Theorem and proved:

Theorem 2.7 (Amoroso - Viada). *Let $\alpha_1, \dots, \alpha_n$ be multiplicatively independent algebraic numbers in a number field K of degree $D = [K : \mathbb{Q}]$. Then*

$$h(\alpha_1) \dots h(\alpha_n) \geq \frac{1}{D} \frac{1}{(1050\, n^5 \mathrm{Log}\,(3D))^{n^2(n+1)^2}}. \tag{2.21}$$

The assumption of being multiplicatively independent was reconsidered in the multiplicative group $\overline{\mathbb{Q}}^{\times}/\mathrm{Tor}(\overline{\mathbb{Q}}^{\times})$ by Vaaler in [Var].

Let A/K be an abelian variety over K a number field. Let $\mathscr{L}$ be a line bundle over A. Let V be a subvariety of A defined over K. The degree $\deg_{\mathscr{L}}(V)$ of V relatively to the Cartier divisor D associated with $\mathscr{L}$ is defined by the theory of intersection [Rz3]. In particular, if $P \in A(\overline{K})$, and $V = \overline{\{P\}}$, then $\deg_{\mathscr{L}}(V) = [K(P) : K]$.

For any $P \in A(\overline{K})$ the *obstruction index* $\delta_{K,\mathscr{L}}(P)$ of P is now extended as

$$:= \min\{\deg_{\mathscr{L}}(V)^{\frac{1}{\operatorname{codim}(V)}} \mid V_{/K} \text{ subvariety of } A, \text{for which } P \in V(\overline{K})\}.$$

Conjecture 9. *(David-Hindry, 2000) (Abelian Lehmer problem) Let A/K be an abelian variety over a number field K and $\mathscr{L}$ an ample symmetric line bundle over A. Then there exists a real number $c(A,K,\mathscr{L}) > 0$ such that the canonical height $\widehat{h}_{\mathscr{L}}(P)$ of P satisfies*

$$\widehat{h}_{\mathscr{L}}(P) \geq \frac{c(A,K,\mathscr{L})}{\delta_{K,\mathscr{L}}(P)} \tag{2.22}$$

for every point $P \in A(\overline{K})$ of infinite order modulo every proper abelian subvariety $V_{/K}$ of A. Moreover, if $D = [K(P) : K]$, for any $P \in A(\overline{K})$ not being in the torsion,

$$\widehat{h}_{\mathscr{L}}(P) \geq \frac{c(A,K,\mathscr{L})}{D^{1/g_0}} \tag{2.23}$$

where g_0 is the dimension of the smallest algebraic subgroup containing P.

For any abelian variety A defined over a number field K [Hy], let us denote, for any integer $n \geq 1$, $K_n := K(A[n])$ the extension generated by the group of the torsion points $A[n]$, so that $K_{\text{tors}} = \cup_{n\geq 1} K(A[n])$.

Theorem 2.8 (Ratazzi [Rz5]). *Let A/K be a CM abelian variety of dimension g over a number field K and $\mathscr{L}$ an symmetric ample line bundle over A. Then there exists a real number $c(A,K,\mathscr{L}) > 0$ such that, for every point $P \in A(\overline{K})$, the canonical height $\widehat{h}_{\mathscr{L}}(P)$ satisfies either*

$$(i) \qquad \widehat{h}_{\mathscr{L}}(P) \geq \frac{c(A/K,\mathscr{L})}{\delta_{K_n,\mathscr{L}}(P)} \left(\frac{\operatorname{Log}\operatorname{Log} 3\,[K_n : K]\,\delta_{K_n,\mathscr{L}}(P)}{\operatorname{Log} 2\,[K_n : K]\,\delta_{K_n,\mathscr{L}}(P)}\right)^{\eta(g)} \tag{2.24}$$

with $\eta(g) = (2g+5)(g+2)(g+1)!(2g.g!)^g$; or

(ii) the point P belongs to a proper torsion subvariety, $B \subset A_{K_n}$, defined over K_n, having a degree bounded by

$$(\deg_{\mathscr{L}} B)^{1/\operatorname{codim} B} \leq \frac{1}{c(A/K,\mathscr{L})} \delta_{K_n,\mathscr{L}}(P) \left(\operatorname{Log} 2[K_n : K]\delta_{K_n,\mathscr{L}}(P)\right)^{2g+2\eta(g)}.$$

(ii) *The relative Lehmer problem*: the generalization of the classical Lehmer problem for subfields $K \subset \overline{\mathbb{Q}}$ is decomposed into two steps:

(ii-i) does there exist a real number $c(K) > 0$ such that $h(\alpha) \geq c(K)$ for all $\alpha \in \mathbb{G}_m(K)/\mathbb{G}_m(K)_{\text{tors}}$?

(ii-ii) if (i) is satisfied, does there exist a real number $c'(K) > 0$ such that, for all $\alpha \in \mathbb{G}_m(\overline{K})/\mathbb{G}_m(\overline{K})_{\text{tors}}$, $h(\alpha) \geq \frac{c'(K)}{[K(\alpha):K]}$?

If K is a number field, (ii-i) is satisfied by Northcott's Theorem and (ii-ii) amounts to the classical Lehmer problem. If K is an infinite extension of $\mathbb{Q}$ the problem is more difficult. In (ii-i), when the field K is $\mathbb{Q}^{\text{ab}}$, or the abelian closure of a number field, it is usual to speak of the *abelian Lehmer problem*. The abelian Lehmer problem was solved by Amoroso and Dvornicich [AD]: they proved that, if $\mathbb{L}/\mathbb{Q}$ is an abelian extension of number fields,

$$h(\alpha) \geq \frac{\operatorname{Log} 5}{12}$$

for any nonzero $\alpha \in \mathbb{L}$ which is not a root of unity. As for (ii-ii), it is usual to speak of *relative Lehmer problem*. The abelian and the relative Lehmer problems are naturally extended in higher dimension. If G denotes either an abelian variety A/K over a number field K or the n-torus $\mathbb{G}_m^n$, and $K_{\text{tors}} = K(G_{\text{tors}})$, the minorant function of the height is expected to depend upon the "nonabelian part of the degree D", where $D = [K(P) : K]$. This "nonabelian part : $D_{\text{tors}} = [K_{\text{tors}}(P) : K_{\text{tors}}]$ of D" is equal to $[K^{\text{ab}}(P) : K^{\text{ab}}]$, where K^{ab} is the abelian closure of K (if $G = A$, A is assumed CM).

Given an abelian extension $\mathbb{L}/\mathbb{K}$ of number fields and a nonzero algebraic number α which is not a root of unity, with $D := [\mathbb{L}(\alpha) : \mathbb{L}]$, Amoroso and Zannier [AZ] proved the following result, which makes use of Dobrowolski's minoration and the previous minoration:

$$h(\alpha) \geq \frac{c(\mathbb{K})}{D} \left(\frac{\operatorname{Log}\operatorname{Log} 5D}{\operatorname{Log} 2D} \right)^{13},$$

where $c(\mathbb{K}) > 0$, in the direction of the relative problem. Amoroso and Delsinne [ADn] computed a lower bound, depending upon the degree and the discriminant of the number field $\mathbb{K}$, for the constant $c(\mathbb{K})$. In 2010, given $\mathbb{K}/\mathbb{Q}$ an extension of algebraic number fields, of degree d, Amoroso and Zannier [AZ2] showed

$$h(\alpha) \geq 3^{-d^2-2d-6}$$

for any nonzero algebraic number α which is not a root of unity such that $\mathbb{K}(\alpha)/\mathbb{K}$ is abelian. As a corollary they obtained

$$h(\alpha) \geq 3^{-14}$$

for any dihedral extension $\mathbb{L}/\mathbb{Q}$ and any nonzero $\alpha \in \mathbb{L}$ which is not a root of unity. For cyclotomic extensions, they obtained sharper results: (i) if $\mathbb{K}$ is a number field of degree d, there exists an absolute constant $c_2 > 0$ such that, with $\mathbb{L}$ denoting the number field generated by $\mathbb{K}$ and any given root of unity, then

$$h(\alpha) \geq \frac{c_2}{d} \frac{(\operatorname{Log}\operatorname{Log} 5d)^3}{(\operatorname{Log} 2d)^4},$$

for any nonzero $\alpha \in \mathbb{L}$ which is not a root of unity; (ii) if $\mathbb{K}$ is a number field of degree d, and α any nonzero algebraic number, not a root of unity, such that $\alpha^n \in \mathbb{K}$ for some integer n under the assumption that $\mathbb{K}(\alpha)/\mathbb{K}$ is an abelian extension, then

$$h(\alpha) \geq \frac{c_3}{d} \frac{(\mathrm{Log}\,\mathrm{Log}\,5d)^2}{(\mathrm{Log}\,2d)^4},$$

for some constant $c_3 > 0$.

In higher dimension [Did] [Rz3], with $G = A$ an abelian variety over a number field K, the *torsion obstruction index* $\delta^{\mathrm{tors}}_{K,\mathscr{L}}(P)$ of a point P is now defined by

$$:= \min\{\deg_{\mathscr{L}^{\mathrm{tors}}}(V)^{\frac{1}{\mathrm{codim}(V)}} \mid V_{/K_{\mathrm{tors}}} \text{ subvariety of } A_{K_{\mathrm{tors}}}, \text{for which } P \in V(\overline{K})\}.$$

Conjecture 10. *(David) Let A/K be an abelian variety over a number field K and $\mathscr{L}$ an ample symmetric line bundle over A. Then there exists a real number $c(A,K,\mathscr{L}) > 0$ such that the canonical height $\widehat{h}_{\mathscr{L}}(P)$ satisfies*

$$\widehat{h}_{\mathscr{L}}(P) \geq \frac{c(A,K,\mathscr{L})}{\delta^{\mathrm{tors}}_{K,\mathscr{L}}(P)} \tag{2.25}$$

for every point $P \in A(\overline{K})$ of infinite order modulo every proper abelian subvariety $V_{/K}$ of A.

The analogue of Amoroso and Dvornicich's theorem [AD] (abelian Lehmer problem) was obtained by Baker and Silverman for abelian varieties [Bk] [BkSn] and for elliptic curves by Baker [Bk], then by Silverman [Sn4]:

Theorem 2.9 (Baker - Silverman). *Let A/K be an abelian variety over a number field K and $\mathscr{L}$ an symmetric ample line bundle over A. Let $\widehat{h}(P) : A(\overline{K}) \to \mathbb{R}$ the associated canonical height. Then there exists a real number $c(A,K,\mathscr{L}) > 0$ such that*

$$\widehat{h}(P) \geq c(A,K,\mathscr{L}) \qquad \text{for all nontorsion points } P \in A(K^{\mathrm{ab}}). \tag{2.26}$$

The proof relies upon Zahrin's theorem on torsion points of abelian varieties deduced from the proof of Faltings's theorem [Fgs] of the Mordell Conjecture.

Theorem 2.10 (Silverman). *Let $K/\mathbb{Q}$ be a number field, let E/K be an elliptic curve, and $\widehat{h} : E(\overline{K}) \to \mathbb{R}$ be the canonical height on E. There is a constant $C(E/K) > 0$ such that every nontorsion point $P \in E(K^{\mathrm{ab}})$ satisfies*

$$\widehat{h}(P) > C(E/K).$$

Small points were studied by Carrizosa [Cza2]. Ratazzi [Rz] obtained the relative version of Amoroso and Zannier's minoration [AZ]:

Theorem 2.11 (Ratazzi). *Let E/K be an elliptic curve with complex multiplication over a number field K. Then there exists a constant $c(E,K) > 0$ such that*

$$\widehat{h}(P) \geq \frac{c(E,K)}{D}\left(\frac{\operatorname{Log}\operatorname{Log} 5D}{\operatorname{Log} 2D}\right)^{13} \qquad \text{for all nontorsion points } P \in E(\overline{K}), \quad (2.27)$$

where $D = [K^{\mathrm{ab}}(P) : K^{\mathrm{ab}}]$.

In the direction of the relative problem, a better lower bound of the canonical height of a point P in a CM abelian variety A/K in terms of the degree of the field generated by P over $K(A_{\mathrm{tors}})$ was obtained by Carrizosa [Cza]. For tori Delsinne [Dle] obtained the following (the obstruction index $\omega_K(\alpha)$ is defined below):

Theorem 2.12 (Delsinne). *Let $n \geq 1$ be an integer. There exist constants $c_1(n)$, $\kappa_1(n)$, $\mu(n)$, $\eta_1(n) > 0$ such that, for any $\alpha \in \mathbb{G}_m^n(\overline{\mathbb{Q}})$ satisfying*

$$h(\alpha) \leq \left(c_1(n)\omega_{\mathbb{Q}^{\mathrm{ab}}}(\alpha)\big(\operatorname{Log}(3\omega_{\mathbb{Q}^{\mathrm{ab}}}(\alpha))\big)^{\kappa_1(n)}\right)^{-1},$$

there exists a torsion subvariety B containing α, the degree of B being bounded by

$$(\deg B)^{1/\operatorname{codim} B} \leq c_1(n)\omega_{\mathbb{Q}^{\mathrm{ab}}}(\alpha)^{\eta_1(n)}\big(\operatorname{Log}(3\omega_{\mathbb{Q}^{\mathrm{ab}}}(\alpha))\big)^{\mu(n)};$$

the constants are effective and one can take the following values:

$$c_1(n) = \exp\Big(64nn!(2(n+1)^2(n+1)!)^{2n}\Big),$$

$$\kappa_1(n) = 3(2(n+1)^2(n+1)!)^n, \quad \mu(n) = 8n!(2(n+1)^2(n+1)!)^n,$$

$$\eta_1(n) = (n-1)!\left(\sum_{i=0}^{n-3}\frac{1}{i!} + 1\right)$$

(iii) *Lehmer's problem for subvarieties*: The extension from points to subvarieties has been formulated for nontorsion subvarieties V of the multiplicative group $\mathbb{G}_m^n$ or of an abelian variety A/K over a number field K by David and Philippon [DPn] [DPn2] and Ratazzi [Rz2] [Rz3]. The natural extension of the minoration problem for the height consists in obtaining the best minoration of *the height* $\widehat{h}_{\mathscr{L}}(V)$, resp. of the essential minimum, as a function of the degree of V or of the *obstruction index of* V. The *obstruction index* $\delta_{K,\mathscr{L}}(V)$ *of* V, resp. $\omega_K(V)$, extends the obstruction index $\delta_{K,\mathscr{L}}(P)$ of a point P [DPn2]. As for the definition of the height of V relatively to an symmetric ample line bundle $\mathscr{L}$, two approaches were followed [Rz3]: one by Philippon [Phn], another one by Bost, Gillet and Soulé [BGS], using theorems of Soulé [Sle] and Zhang [Zhg2]. In the second construction Zhang [Zhg2] showed how to consider the canonical height (or Néron-Tate height, or normalized height) $\widehat{h}_{\mathscr{L}}(V)$ as a limit of Arakelov heights.

Define the canonical height (say) $\widehat{h}$ on $\mathbb{G}_m^n(\overline{\mathbb{Q}})$ by $\widehat{h}(\alpha_1,\ldots,\alpha_n) = h(\alpha_1)+\ldots+h(\alpha_n)$. For $\theta > 0$, let V be a subvariety of $\mathbb{G}_m^n$ defined over $\overline{\mathbb{Q}}$. For $\theta > 0$, let:

$$V_\theta := \{P \in V(\overline{\mathbb{Q}}) \mid \leq \theta\},$$

and the *essential minimum*

$$\widehat{\mu}^{\text{ess}}(V) := \inf\{\, > 0 \mid V_\theta \text{ is Zariski dense in } V\}.$$

The generalized Bogomolov conjecture for subvarieties of tori asserts that $\widehat{\mu}^{\text{ess}}(V) = 0$ is and only if V is a torsion subvariety. In the case where V is a point, $V = \{P\}$, $\widehat{\mu}^{\text{ess}}(V) = \widehat{h}(P)$. Zhang [Zhg] [Zhg2] [Zhg3] showed that the minoration problem of $\widehat{\mu}^{\text{ess}}(V)$ is essentially the same problem as finding lower bounds for the canonical height $\widehat{h}(V)$ of V, in the sense of Arakelov theory. Indeed, from his Theorem of the Successive Minima, Zhang proved:

$$\widehat{\mu}^{\text{ess}}(V) \leq \frac{\widehat{h}(V)}{\deg(V)} \leq (\dim(V)+1)\,\widehat{\mu}^{\text{ess}}(V)$$

for V any subvariety of $\mathbb{G}_m^n$ over $\overline{\mathbb{Q}}$. Zhang obtained similar results for subvarieties of abelian varieties. The canonical height $\widehat{h}(V)$ of V is related to the problem of minoration of multivariate Mahler measures by the following: for V being a hypersurface defined by a polynomial $F(x_1,\ldots,x_n) \in \mathbb{Z}[x_1,\ldots,x_n]$ (having relatively prime integer coefficients), then

$$\widehat{h}(V) = \int_0^1 \cdots \int_0^1 \operatorname{Log}|F(e^{2\pi i t_1},\ldots,e^{2\pi i t_n})|dt_1\ldots dt_n$$

is the logarithmic Mahler measure $\operatorname{Log}\mathrm{M}(F)$ of F. Let K be a field of characteristic zero, and let V be a subvariety of $\mathbb{G}_m^n$ defined over $\overline{\mathbb{Q}}$. Define the index of obstruction $\omega_K(V)$ to be the minimum degree of a nonzero polynomial $F \in K[x_1,\ldots,x_n]$ vanishing identically on V. Equivalently, it is the minimum degree of a hypersurface defined over K and containing V. The *higher-dimensional Lehmer Conjecture* takes the following form (i.e. the two following conjectures):

Conjecture 11. *(Amoroso - David, 1999) Let V be a subvariety of $\mathbb{G}_m^n$, and assume that V is not contained in any torsion subvariety (i.e., a translate of a proper subgroup by a torsion point). Then there exists a constant $C(n) > 0$ such that*

$$\widehat{\mu}^{\text{ess}}(V) \geq \frac{C(n)}{\omega_{\mathbb{Q}}(V)}.$$

A 0-dimensional subvariety $V = (\alpha_1,\ldots,\alpha_n)$ of $\mathbb{G}_m^n$ is contained in a torsion subvariety if and only if $\alpha_1,\ldots,\alpha_n$ are multiplicatively dependent.

In a similar way, for $\theta > 0$, V a subvariety of an abelian variety A defined over a number field K, and $\mathscr{L}$ a symmetric ample line bundle on A, we define: $V(\theta,\mathscr{L}) :=$

$\{x \in V(K) \mid \widehat{h}_{\mathscr{L}}(\overline{K}) \leq \theta\}$. The *essential minimum* of V is

$$\widehat{\mu}^{\text{ess}}_{\mathscr{L}}(V) := \{\theta > 0 \mid \overline{V(\theta, \mathscr{L})} = V\}$$

where $\overline{V(\theta, \mathscr{L})}$ is the adherence of Zariski of $V(\theta, \mathscr{L})$ in A.

Conjecture 12. *(David - Philippon, 1996) Let A be an abelian variety defined over a number field K, and $\mathscr{L}$ a symmetric ample line bundle on A. Let V/K be a proper subvariety of A, K-irreducible and such that $V_{\overline{K}}$ is not the union of torsion subvarieties, then*

$$\frac{\widehat{h}_{\mathscr{L}}(V)}{\deg_{\mathscr{L}}(V)} \geq \frac{c(A/K, \mathscr{L})}{(\deg_{\mathscr{L}}(V))^{1/(s-\dim(V))}}$$

for some constant $c(A/K, \mathscr{L}) > 0$ depending on A/K and $\mathscr{L}$, where s is the dimension of the smallest algebraic subgroup containing V.

Generalizing (2.20) the *higher dimensional Dobrowolski bound* takes the following form, proved in [ADd2] for $\dim(V) = 0$, in [ADd3] for $\text{codim}(V) = 1$ and in [ADd4] for varieties of arbitrary dimension.

Theorem 2.13 (Amoroso - David). *Let V be a subvariety of $\mathbb{G}_m^n$ defined over $\mathbb{Q}$ of codimension k. Let us assume that V is not contained in any union of proper torsion varieties. Then, there exist two constants $c(n)$ and $\kappa(n) = (k+1)(k+1)!^k - k$ such that*

$$\widehat{\mu}^{\text{ess}}(V) \geq \frac{C(n)}{\omega_{\mathbb{Q}}(V)} \frac{1}{(\text{Log}\, 3\omega_{\mathbb{Q}}(V))^{\kappa(k)}}.$$

Amoroso and Viada [AVa] introduced relevant invariants of a proper projective subvariety $V \subset \mathbb{P}^n$: e.g. $\delta(V)$ defined as the minimal degree δ such that V is, as a set, the intersection of hypersurfaces of degree $\leq \delta$.

Theorem 2.14 (Amoroso - Viada [AVa2]). *Let $V \subset \mathbb{G}_m^n$ be a $\mathbb{Q}$-irreducible variety of dimension d. Then, for any $\alpha \in V^*(\overline{\mathbb{Q}})$,*

$$h(\alpha) \geq \frac{1}{\delta(V)} \frac{1}{(935\, n^5 \text{Log}\,(n^2 \delta(V))^{(d+1)(n+1)^2}}.$$

Following the main Theorem1.3 in [AVa2] the essential minimum admits the following lower bound:

Theorem 2.15 (Amoroso - Viada). *Let $V \subset \mathbb{G}_m^n$ be a $\mathbb{Q}$-irreducible variety of dimension k which is not contained in any union of proper torsion varieties. Then,*

$$\widehat{\mu}^{\text{ess}}(V) \geq \frac{1}{\omega_{\mathbb{Q}}(V)} \frac{1}{(935\, n^5 \text{Log}\,(n^2 \omega_{\mathbb{Q}}(V))^{k(k+1)(n+1)}}.$$

Theorem 2.16 (Ratazzi [Rz3]). *Let A be a CM abelian variety defined over a number field K, and $\mathscr{L}$ a symmetric ample line bundle on A. Let V/K be a proper subvariety of A, K-irreducible and such that $V_{\overline{K}}$ is not the union of torsion subvarieties. Then*

$$\frac{\widehat{h}_{\mathscr{L}}(V)}{\deg_{\mathscr{L}}(V)} \geq \widehat{\mu}^{\text{ess}}_{\mathscr{L}}(V) \geq \frac{c(A/K,\mathscr{L})}{(\deg_{\mathscr{L}}(V))^{1/(n-\dim(V)}} \frac{1}{(\mathrm{Log}\,(2\deg_{\mathscr{L}}(V))^{\kappa(n)}}$$

with $\kappa(n) = (2n(n+1)!)^{n+2}$, for some constant $c(A/K,\mathscr{L}) > 0$ depending only on A/K and $\mathscr{L}$.

Ratazzi in [Rz3] obtained more precise minorations of $\widehat{h}_{\mathscr{L}}(V)$ in the case where V is an hypersurface. In [Rz2] Ratazzi proves that the optimal lower bound given by David and Philippon [DPn] in Conjecture 12 is a consequence of a Conjecture of David and Hindry on the abelian Lehmer problem.

On the way of proving the relative abelian Lehmer Conjecture, Carrizosa [Cza] [Cza3] obtained a lower bound of the canonical height of a point P in a CM abelian variety A/K defined over a number field K in terms of the degree of the field generated by P over $K(A_{\text{tors}})$. As Corollary of Theorem 2.12, with the same constants, Delsinne obtained the relative result:

Theorem 2.17 (Delsinne). *Let V be a subvariety of $\mathbb{G}_m^n$ which is not contained in any proper algebraic subgroup of $\mathbb{G}_m^n$. Then*

$$\widehat{\mu}^{\text{ess}}(V) \geq \Big(c_3(n)\omega_{\mathbb{Q}^{\text{ab}}}(V)(\mathrm{Log}\,(3\omega_{\mathbb{Q}^{\text{ab}}}(V)))^{\kappa_1(n)}\Big)^{-1}$$

with $c_3(n) = c_1(n)(\dim(V)+1)$.

Concomitantly to the Lehmer problems, the geometry of the distribution of the small points, their Galois orbits, the limit equidistribution of conjugates on some subvarieties, the theorems of finiteness, were considered e.g. in Amoroso and David [ADd6] [ADd7] Bilu [Bu], Bombieri [Bri], Burgos Gil, Philippon, Rivera-Letelier and Sombra [BGPRLS], Chambert-Loir [CLr], D'Andrea, Galligo, Narváez-Clauss and Sombra [DGS] [DNS], Favre and Rivera-Letelier [FLr], Habegger [Hgr], Hughes and Nikeghbali [HNi], Litcanu [Ltu], Petsche [Pe] [Pe2], Pritsker [Pr3], Ratazzi and Ullmo [RzU], Rémond [Rd], Rumely [Rly], Szpiro, Ullmo and Zhang [SUZ], Zhang [Zhg2] [Zhg3].

The type of proof of Dobrowolski [Do2], even revisited or generalized (Amoroso and David [ADd], Carrizosa [Cza3], Laurent [La], Meyer [Me], Ratazzi [Rz4]), leads to weaker minorations of the height than the better ones obtained by means of the dynamical zeta function of the β-shift, as in Theorem 1.4, compared to (1.14), for the classical case.

2.3 Analogues of the Mahler measure and Lehmer's problem

Several generalizations and analogues of the Mahler measure were introduced, for which the analogue of the problem of Lehmer holds, or not.

The *Zhang-Zagier height* $\mathscr{H}(\alpha)$ of an algebraic number α is defined as $\mathscr{H}(\alpha) = \mathscr{M}(\alpha)\mathscr{M}(1-\alpha)$. After Zhang [Zhg] and Zagier [Za] [Za2], if α is an algebraic number different from the roots of $(z^2-z)(z^2-z+1)$, then

$$\mathscr{H}(\alpha) \geq \sqrt{\frac{1+\sqrt{5}}{2}} = 1.2720196\ldots$$

Doche [Dhe] [Dhe2], using (2.9), obtains the following better minorant: if α is an algebraic number different from the roots of $(z^2-z)(z^2-z+1)\Phi_{10}(z)\Phi_{10}(1-z)$, then

$$\mathscr{H}(\alpha) \geq 1.2817770214 =: \eta, \tag{2.28}$$

and the smallest limit point of $\{\mathscr{H}(\alpha) \mid \alpha \in \overline{\mathbb{Q}}\}$ lies in $[1.2817770214, 1.289735]$. The lower bound η is used in Proposition 3.15, relative to the height one trinomials G_n.

Dresden [Ddn] introduced a *generalization of the Zhang-Zagier height*: given G a subgroup of $PSL(2,\overline{\mathbb{Q}})$, the G-orbit height of $\alpha \in \mathbb{P}^1(\overline{\mathbb{Q}})$ is

$$h_G(\alpha) := \sum_{g\in G} h(g\alpha).$$

For G the cyclic group generated by

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

Dresden finds, for $\alpha \neq 0, \neq 1$ not being a primitive sixth root of unity,

$$h(\alpha) + h(\frac{1}{1-\alpha}) + h(\frac{1}{\alpha}) \geq 0.42179\ldots$$

with equality for α any root of $(X^2-X+1)^3 - X^2(X-1)^2$; otherwise, $h_G(\alpha) = 0$.

The *G-invariant Lehmer problem* is stated as follows in van Ittersum ([Itm] p. 146): given G a finite subgroup of $PSL(2,\mathbb{Q})$, does there exist a positive constant $D = D_G > 0$ such that

$$h_G(\alpha) = 0 \quad \text{or} \quad h_G(\alpha) \geq D, \qquad \text{for all } \alpha \in \mathbb{P}^1(\overline{\mathbb{Q}})?$$

If G is trivial this constant D does not exist [Za]. Denote by Orb_G the set of all orbits of the action of G on $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and $\mathrm{Orb}_{G,unit} := \{Y \in \mathrm{Orb}_G \mid \text{for all } \alpha \in Y, \alpha = 0 \text{ or } |\alpha| = 1\}$. Dresden's result [Ddn] was generalized in [Itm]: van Ittersum [Itm] proved the G-invariant Lehmer problem under the assumption on G that $\mathrm{Orb}_{G,unit}$ is finite.

The *(logarithmic) metric Mahler measure* $\widehat{m} : \mathscr{G} \to [0,\infty)$ was introduced by Dubickas and Smyth in [DsSy2] [DsSy3], where

$$\mathscr{G} := \overline{\mathbb{Q}}^{\times} / \mathrm{Tor}(\overline{\mathbb{Q}}^{\times})$$

is the $\mathbb{Q}$-vector space of algebraic numbers modulo torsion, written multiplicatively. For $\underline{\alpha} \in \mathscr{G}$ it is defined by

$$\widehat{m}(\underline{\alpha}) := \inf\Big\{ \sum_{n=1}^{N} \mathrm{Log}\,\mathrm{M}(\alpha_n) \mid N \in \mathbb{N},\, \alpha_n \in \overline{\mathbb{Q}}^{\times},\, \alpha = \prod_{n=1}^{N} \alpha_n \Big\}$$

where the infimum is taken over all possible ways of writing any representative α of $\underline{\alpha}$ as a product of other algebraic numbers. The construction may be applied to any height function [DsSy3] and is extremal in the sense that any other function $g : \mathscr{G} \to [0,\infty)$ satisfying (i) $g(\underline{\alpha}) \leq \widehat{m}(\underline{\alpha})$ for any $\underline{\alpha} \in \mathscr{G}$, (ii) $g(\underline{\alpha}\,\underline{\beta}^{-1}) \leq g(\underline{\alpha}) + g(\underline{\beta})$ for any $\underline{\alpha}, \underline{\beta} \in \mathscr{G}$ (triangle inequality), is smaller than $\widehat{m}$.

The structure of the completion of $\mathscr{G}$, as a Banach space over the field $\mathbb{R}$ of real numbers, endowed with the norm deduced from the Weil height has been studied by Allcock and Vaaler [AkV]. Indeed, by construction, the Weil height satisfies: for any $\alpha \in \overline{\mathbb{Q}}^{\times}$ and any root of unity ζ, $h(\alpha) = h(\zeta\alpha)$, so that h extends to $h : \mathscr{G} \to \infty$ with the properties:

(i) $h(\underline{\alpha}) = 0$ if and only if $\underline{\alpha}$ is the identity element $\underline{1}$ in $\mathscr{G}$,
(ii) $h(\underline{\alpha}) = h(\underline{\alpha}^{-1})$ for all $\underline{\alpha} \in \mathscr{G}$,
(iii) $h(\underline{\alpha}\,\underline{\beta}) \leq h(\underline{\alpha}) + h(\underline{\beta})$ for all $\underline{\alpha}, \underline{\beta} \in \mathscr{G}$.

These conditions imply that the map $(\underline{\alpha}, \underline{\beta}) \to h(\underline{\alpha}\,\underline{\beta}^{-1})$ is a metric on the quotient group $\mathscr{G}$, on which the $\mathbb{Q}$-action is defined by $(r/s, \underline{\alpha}) \to \underline{\alpha}^{r/s}$ by the roots of the polynomials $z^s - (\zeta\alpha)^r = 0$ for any $\alpha \in \overline{\mathbb{Q}}^{\times}$ and any root ζ of unity. With the usual absolute value $|\cdot|$ on $\mathbb{Q}$, $h(\alpha^{r/s}) = |\frac{r}{s}| h(\alpha)$, and h is a norm on the $\mathbb{Q}$-vector space $\mathscr{G}$.

Let Y denote the totally disconnected, locally compact, Hausdorff space of all places y of $\overline{\mathbb{Q}}$. Let $\mathscr{B}$ be the Borel σ-algebra of Y. For any number field $k \subset \overline{\mathbb{Q}}$ such that $k/\mathbb{Q}$ is Galois and any place v of k, denote $Y(k,v) := \{y \in Y \mid y|v\}$ so that

$$Y = \bigsqcup_{\mathrm{all places}\, v\, \mathrm{of}\, k} Y(k,v) \qquad (\text{disjoint union}).$$

Let λ be the unique regular measure on $\mathscr{B}$, positive on open sets, finite on compact sets, which satisfies:

$$(i) \quad \lambda(Y(k,v)) = \frac{[k_v : \mathbb{Q}_v]}{[k : \mathbb{Q}]} \quad \text{for any Galois } k/\mathbb{Q}, \text{ any place } v \text{ of } k,$$

(ii) $\lambda(\tau E) = \lambda(E)$ for all $\tau \in \mathrm{Aut}(\overline{\mathbb{Q}}/k)$ and $E \in \mathscr{B}$. Allcock and Vaaler [AkV] proved that the (not surjective) map

$$f : \mathscr{G} \to L^1(Y, \mathscr{B}, \lambda), \;\; \alpha \to f_\alpha \text{ given by } \; f_\alpha(y) := \mathrm{Log}\,\|\alpha\|_y$$

is a linear isometry of norm $2h$, i.e. $f_{\alpha\beta}(y) = f_\alpha(y) + f_\beta(y)$, $f_{\alpha^{r/s}}(y) = (r/s) f_\alpha(y)$, $\int_Y |f_\alpha(y)| d\lambda(y) = 2h(\alpha)$, with the property: $\int_Y f_\alpha(y) d\lambda(y) = 0$. Denote by $\mathscr{F} := f(\mathscr{G})$ the image of $\mathscr{G}$ in $L^1(Y,\mathscr{B},\lambda)$ and $\chi := \{F \in L^1(Y,\mathscr{B},\lambda) \mid \int_Y F(y) d\lambda(y) = 0\}$ the co-dimension one linear subspace of $L^1(Y,\mathscr{B},\lambda)$. They proved that $\mathscr{F}$ is dense in χ ([AkV] Theorem 1), i.e. that χ is the completion of $(\mathscr{G},h)$, up to isometry. They also proved that, for any real $1 < p < \infty$, $\mathscr{F}$ is dense in $L^p(Y,\mathscr{B},\lambda)$ ([AkV] Theorem 2), and $\mathscr{F}$ is dense in the Banach space $\mathscr{C}_0(Y)$ of continuous real valued functions on Y which vanish at infinity, equipped with the sup-norm ([AkV] Theorem 3).

Fili and Miner [FMr2] proved that the space $\mathscr{F}$ admits linear operators canonically associated to the Mahler measure and to the L^p norms on Y. They introduced norms, called *Mahler p-norms*, from orthogonal decompositions of $\mathscr{F}$, and, in this context, obtained extended formulations, called *L^p Lehmer Conjectures*, of the Lehmer Conjecture and the Conjecture of Schinzel-Zassenhaus. Namely, let $\mathscr{K}$ be the set of finite extensions of $\mathbb{Q}$ and $\mathscr{K}^G := \{K \in \mathscr{K} \mid \sigma(K) = K \text{ for all } \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\}$. For each $K \in \mathscr{K}$, denote by $V_K := \{f_\alpha \mid \alpha \in K^\times/\mathrm{Tor}(K^\times)\}$ the $\mathbb{Q}$-vector subspace of $\mathscr{F}$ constituted by the nonzero elements of K modulo torsion, and, for $n \geq 0$, $V^{(n)} := \sum_{K \in \mathscr{K}, [K:\mathbb{Q}] \leq n} V_K$. Denote by $\langle f, g \rangle = \int_Y f(y) g(y) d\lambda(y)$ the inner product on $\mathscr{F}$.

Theorem 2.18 (Fili - Miner). *(i) There exist projection operators $T_K : \mathscr{F} \to \mathscr{F}$ for each $K \in \mathscr{K}^G$ such that $T_K(\mathscr{F}) \subset V_K$, $T_K(\mathscr{F}) \perp T_L(\mathscr{F})$ for all $K, L \in \mathscr{K}^G$, $K \neq L$, with respect to the inner product on $\mathscr{F}$, and*

$$\mathscr{F} = \bigoplus_{K \in \mathscr{K}^G} T_K(\mathscr{F}),$$

(ii) for all $n \geq 1$, there exist projections $T^{(n)} : \mathscr{F} \to \mathscr{F}$ such that $T^{(n)}(\mathscr{F}) \subset V^{(n)}$, $T^{(m)}(\mathscr{F}) \perp T^{(n)}(\mathscr{F})$ for all $m \neq n$, and

$$\mathscr{F} = \bigoplus_{K \in \mathscr{K}^G} T^{(n)}(\mathscr{F}),$$

(iii) for every $K \in \mathscr{K}^G$ and $n \geq 0$, the projections T_K and $T^{(n)}$ commute.

Now, for any $\alpha \in \overline{\mathbb{Q}}^\times$ and any real number $1 \leq p \leq \infty$, let $h_p(\alpha) := \|f_\alpha\|_p$ (recalling that $h_1(\alpha) = 2h(\alpha)$).

Conjecture 13. *(Fili - Miner)(L^p Lehmer Conjectures) For any real number $1 \leq p \leq \infty$, there exists a real constant $c_p > 0$ such that*

$$(*_p) \qquad m_p(\alpha) := \deg_{\mathbb{Q}}(\alpha)\, h_p(\alpha) \geq c_p \qquad \text{for all } \alpha \in \overline{\mathbb{Q}} \setminus \mathrm{Tor}(\overline{\mathbb{Q}}).$$

For $p = 1$ Conjecture 13 is exactly the classical Lehmer Conjecture. Moreover, Fili and Miner ([FMr2], Proposition 4.1) proved that, for $p = \infty$, Conjecture 13 is exactly the classical Conjecture of Schinzel-Zassenhaus.

The operator $M : \mathscr{F} \to \mathscr{F}, f \to \sum_{n=1}^{\infty} nT^{(n)} f$ is well-defined, unbounded, invertible, and is always a finite sum. The norm $f \to \|Mf\|_p$ is called the Mahler p-norm on $\mathscr{F}$. For any $f \in \mathscr{F}$, let $d(f) := \min\{\deg_{\mathbb{Q}}(\alpha) \mid \alpha \in \overline{\mathbb{Q}}^{\times}, f_\alpha = f\}$ be the smallest degree possible in the class of f. For any $f \in \mathscr{F}$, the *minimal field*, denoted by K_f, is defined to be the minimal element of the set $\{K \in \mathscr{K} \mid f \in V_K\}$. Let $\delta(f) = [K_f : \mathbb{Q}]$. The P_K operators on $\mathscr{F}$ are defined from the T_K operators as: $P_K := \sum_{F \in \mathscr{K}^G, F \subset K} T_F$. An element $f \in \mathscr{F}$ is said to be *Lehmer irreducible* (or *representable*) if $\delta(f) = d(f)$. The set of Lehmer irreducible elements of $\mathscr{F}$ is denoted by $\mathscr{L}$. An element $f \in \mathscr{F}$ is said to be *projection irreducible* if $P_H(f) = 0$ for all propers subfields H of K_f. The set of projection irreducible elements of $\mathscr{F}$ is denoted by $\mathscr{P}$. Let $\mathscr{U} = \{f \in \mathscr{F} \mid \mathrm{supp}_Y(f) \subset Y(\mathbb{Q}, \infty)\}$ be the subset of algebraic units.

Theorem 2.19 (Fili - Miner). *For every real number $1 \le p \le \infty$, the L^p Lehmer Conjecture $(*_p)$ holds if and only the following minoration on the Mahler p-norms holds*

$$(**_p) \qquad \| \sum_{n=1}^{\infty} nT^{(n)} f_\alpha \|_p \ge c_p \qquad \textit{for all } f_\alpha \in \mathscr{L} \cap \mathscr{P} \cap \mathscr{U}, f_\alpha \neq 0.$$

Further, for $1 \le p \le q \le \infty$, if $(**_p)$ holds, then $(**_q)$ also holds.

An element $f_\beta \in \mathscr{F}$ is said to be a *Pisot number*, resp. a *Salem number*, if it has a representative $\beta \in \overline{\mathbb{Q}}^{\times}$ which is a Pisot number, resp. a Salem number. Fili and Miner ([FMr2], Prop. 4.2 and Prop. 4.3) proved that every Pisot number and every Salem number is Lehmer irreducible, moreover that every Salem number is also projection irreducible. A *surd* is an element $f \in \mathscr{F}$ such that $\delta(f) = 1$, i.e. for which $K_f = \mathbb{Q}$ and $\|Mf\|_p = \|f\|_p = h_p(f)$; a surd is projection irreducible.

The *t-metric Mahler measure*, was introduced by Samuels [JSls] [Sls2] [Sls4]. For $t \ge 1$, the t-metric Mahler measure is defined by

$$\mathrm{M}_t(\alpha) := \inf\Big\{ \Big(\sum_{n=1}^{N} (\mathrm{Log}\, \mathrm{M}(\alpha_n))^t \Big)^{1/t} \mid N \in \mathbb{N}, \alpha_n \in \overline{\mathbb{Q}}^{\times}, \alpha = \prod_{n=1}^{N} \alpha_n \Big\}$$

and, by extension, for $t = \infty$, by

$$\mathrm{M}_\infty(\alpha) := \inf\Big\{ \max_{1 \le n \le N} \{\mathrm{Log}\, \mathrm{M}(\alpha_n)\} \mid N \in \mathbb{N}, \alpha_n \in \overline{\mathbb{Q}}^{\times}, \alpha = \prod_{n=1}^{N} \alpha_n \Big\}.$$

For $t = 1$, M_1 is the metric Mahler measure introduced in [DsSy2]. These functions satisfy an analogue of the triangle inequality [JSls], and the map $(\alpha, \beta) \to \mathrm{M}_t(\alpha\beta^{-1})$ defines a metric on $\mathscr{G} := \overline{\mathbb{Q}}^{\times}/\mathrm{Tor}(\overline{\mathbb{Q}}^{\times})$ which induces the discrete topology if and only if Lehmer's Conjecture is true. For $t \in [1, \infty]$ and $\alpha \in \overline{\mathbb{Q}}$ we say that the infimum in $\mathrm{M}_t(\alpha)$ is attained by $\alpha_1, \ldots, \alpha_n$ if the equality case holds: i.e., for $1 \le t < \infty$, if $\mathrm{M}_t(\alpha) = \big(\sum_{n=1}^{N} (\mathrm{Log}\, \mathrm{M}(\alpha_n))^t\big)^{1/t}$ and, for $t = \infty$, $\mathrm{M}_\infty(\alpha) = \max_{1 \le n \le N} \{\mathrm{Log}\, \mathrm{M}(\alpha_n)\}$. For $\alpha \in \overline{\mathbb{Q}}$, denote by $\mathbb{K}_\alpha$ the Galois closure of $\mathbb{Q}(\alpha)/\mathbb{Q}$, and let $\mathrm{Rad}(\mathbb{K}_\alpha) := \{\beta \in \overline{\mathbb{Q}} \mid$

$\beta^m \in \mathbb{K}_\alpha$ for some $m \in \mathbb{N}\}$. Following a conjecture of Dubickas and Smyth [DsSy2], Samuels [Sls2] [Sls3] proved that the infimum of $\mathrm{M}_t(\alpha)$ is attained in $\mathrm{Rad}(\mathbb{K}_\alpha)$. Whether this infimum is attained in proper subsets of $\overline{\mathbb{Q}}$ leads to many open questions ([JSls], Question 1.5), though Jankauskas and Samuels proved some results for certain cases of decompositions of rational numbers in prime numbers ([JSls], Theorem 1.3, Theorem 1.4). In particular for $\alpha \in \mathbb{Q}$, they proved that the infimum of $\mathrm{M}_t(\alpha)$ may be attained using only rational points.

The p-metric, resp. the t-metric, constructions of Fili and Miner [FMr] and Jankauskas and Samuels [JSls] are of different nature, though they are esentially the same for $p = 1$. Fili and Miner [FMr] studied the minimality of the Mahler measure by several norms, related to the metric Mahler measure introduced in [DsSy2], using results of de la Masa and Friedman [dMaF] on heights of algebraic numbers modulo multiplicative group actions. Fili and Miner [FMr] introduced an infinite collection $(h_t)_t$ of vector space norms on $\mathscr{G}$, called *L^t Weil heights*, $t \in [1,\infty]$, which satisfy extremality properties, and *minimal logarithmic L^t Mahler measures* $(m_t)_t$ from $(h_t)_t$. By definition, for $\mathbb{K}$ a number field, $\Sigma_{\mathbb{K}}$ its set of places and $||.||_\nu$ the absolute value on $\mathbb{K}$ extending the usual p-adic absolute value on $\mathbb{Q}$ if ν is finite or the usual archimedean absolute value if ν is infinite, for $1 \leq t < \infty$ real,

$$h_t(\alpha) := \Big(\sum_{\nu \in \Sigma_{\mathbb{K}}} \frac{[\mathbb{K}_\nu : \mathbb{Q}_\nu]}{[\mathbb{K} : \mathbb{Q}]} . |\mathrm{Log}\,||\alpha||_\nu|^t \Big)^{1/t}, \qquad \alpha \in \mathbb{K}^\times$$

for which $2h = h_1$ [AkV], and

$$h_\infty(\alpha) := \sup_{\nu \in \Sigma_{\mathbb{K}}} |\mathrm{Log}\,||\alpha||_\nu|, \qquad \alpha \in \mathbb{K}^\times.$$

They reformulated Lehmer's Conjecture in this context (with $1 \leq t < \infty$). Because h_∞ serves as a generalization of the (logarithmic) house of an algebraic integer, they also reformulated the Conjecture of Schinzel and Zassenhaus. In [JSls] Jankauskas and Samuels investigate the t-metric Mahler measures of surds and rational numbers.

The *ultrametric Mahler measure* was introduced by Fili and Samuels [FSls] [Sls3] to give a projective height of $\mathscr{G}$, which satisfies the strong triangle inequality. The ultrametric Mahler measure induces the discrete topology on $\mathscr{G}$ if and only if Lehmer's Conjecture is true.

Two p-adic Mahler measures are introduced by Besser and Deninger in [BD] in view of developping natural analogues of the classical logarithmic Mahler measures of Laurent polynomials, following Deninger [Dgr]. The p-adic analogue of Deligne cohomology is now Besser's modified syntomic cohomology, but with the same symbols in the algebraic K-theory groups. For one p-adic Mahler measure the authors show that there is *no analogue* of Lehmer's problem.

Generalized Mahler measures, higher Mahler measures and multiple k-higher Mahler measures were introduced by Gon and Oyanagi [GOi], resp. Kurokawa, Lalín and Ochiai [KLO] and reveal deep connections between zeta functions, polylogarithms, multiple L- functions (Sasaki [Ski]) and multiple sine functions. For any

$n \geq 1$, given $P_1, \ldots, P_s \in \mathbb{C}[x_1, \ldots, x_n]$ (not necessarily distinct) nonzero polynomials, the generalized Mahler measure is defined by $m_{\max}(P_1, \ldots, P_s) :=$

$$\frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \max\{\operatorname{Log}|P_1(x_1, \ldots, x_n)|, \ldots, \operatorname{Log}|P_s(x_1, \ldots, x_n)|\} \frac{dx_1}{x_1} \ldots \frac{dx_n}{x_n},$$

the multiple Mahler measure by $m(P_1, \ldots, P_s) :=$

$$\frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \operatorname{Log}|P_1(x_1, \ldots, x_n)| \ldots \operatorname{Log}|P_s(x_1, \ldots, x_n)| \frac{dx_1}{x_1} \ldots \frac{dx_n}{x_n},$$

the k-higher Mahler measure of P by $m_k(P) := m(P, \ldots, P) =$

$$\frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \operatorname{Log}^k |P_1(x_1, \ldots, x_n)| \frac{dx_1}{x_1} \ldots \frac{dx_n}{x_n},$$

The k-higher Mahler measures are deeply related to the zeta Mahler measures, and their derivatives, introduced by Akatsuka [Aka]. The *problem of Lehmer* for k-higher Mahler measures is considered by Lalín and Sinha in [LinS]. Asymptotic formulas of $m_k(P)$, with k, are given in [Bis] and [LinS], for some families of polynomials P. Analogues of Boyd-Lawton's Theorem are studied in Issa and Lalín [IL]. By analogy with Deninger's approach, the motivic reinterpretation of the values of k-higher Mahler measures in terms of Deligne cohomology is given by Lalín in [Lin3].

The logarithmic Mahler measure m_G *over a compact abelian group* G is introduced by Lind [Ld4]. The group is equipped with the normalized Haar measure μ. By Pontryagin's duality the dual group $\widehat{G}$ (characters) is discrete and the class of functions f to be considered is $\mathbb{Z}[\widehat{G}]$. For $f \in \mathbb{Z}[\widehat{G}]$

$$\mathrm{m}_G(f) = \int_G \operatorname{Log}|f| d\mu \quad \text{generalizes} \quad \mathrm{m}(f) = \int_0^1 \operatorname{Log}|f(e^{2i\pi t})| dt \text{ for } f \in \mathbb{Z}[x^{\pm 1}].$$

The *Lehmer constant of* G is then defined by

$$\lambda(G) := \inf\{\mathrm{m}_G(f) \mid f \in \mathbb{Z}[\widehat{G}], \mathrm{m}_G(f) > 0\}.$$

The author considers several groups G (connected, finite) and the *problem of Lehmer* in each case. The classical Lehmer's problem asks whether $\lambda(\mathbb{T}) = 0$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Let $n \geq 2$, denote by $\rho(n)$ the smallest prime number that does not divide n. Lind proves that $\lambda(G) = \lambda(\mathbb{T})$ for any nontrivial connected compact abelian group, and $\lambda(\mathbb{Z}/n\mathbb{Z}) \leq \frac{\operatorname{Log}\rho(n)}{n}$ for $n \geq 2$. This Lehmer's constant has been named Lind-Lehmer's constant more recently. It is known in some cases [PrVr2]. Kaiblinger [Kgr] obtained results on $\lambda(G)$ for finite cyclic groups G of cardinality not divisible by 420; Pigno and Pinner [PoPr] solved the case $|G| = 420$. De Silva and Pinner [Da] [DaPr] made progress on noncyclic finite abelian groups $G = \mathbb{Z}_p^n$, then Pigno, Pinner and Vipismakul [Vl] [PoPrVl] on general p-groups $G_p = \mathbb{Z}_{p^{l_1}} \times \ldots \times \mathbb{Z}_{p^{l_n}}$ and $G = \mathbb{Z}_m \times G_p$ for m not divisible by p.

An areal analogue of Mahler's measure is reported by Pritsker [Pr], linked to Hardy and Bergman normed spaces of functions on the unit disk.

Lehmer's problems in positive characteristic and Drinfeld modules: let $k = \mathbb{F}_q(T)$ be the fraction field of the ring $\mathbb{F}_q[T]$ of polynomials with coefficients in the finite field $\mathbb{F}_q$ (p is a prime number and q a power of p). Let $k_\infty = \mathbb{F}_q((1/T))$ be the completion of k for the $1/T$-adic valuation v. The valuation, still denoted by v, is extended to the algebraic closure $\overline{k}$ of k, resp. $\overline{k_\infty}$ of k_∞. The degree $\deg(x)$ of $x \in k_\infty$ is equal to the integer-valued $-v(x)$, with the convention $\deg(0) = -\infty$. Let t denote a formal variable. By definition a t-module of dimension N and rank d on $\overline{k}$ is given by the additive group $(\mathbb{G}_a)^N$ and an injective ring homomorphism $\Phi : \mathbb{F}_q[T] \to \mathrm{End}(\mathbb{G}_a)^N$ which satisfies:

$$\Phi(t) = a_0 F^0 + \ldots + a_d F^d,$$

where F is the Frobenius endomorphism on $(\mathbb{G}_a)^N$ and $a_0, a_1, \ldots, a_d$ are $N \times N$ matrices with coefficients in $\overline{k}$. In [Dis] Denis constructed a canonical height $\widehat{h} = \widehat{h}_\Phi$ on t-modules for which a_d is invertible, from the Weil height. Denis formulated Lehmer's problem for t-modules as follows, in two steps: (i) for $\alpha \in (\overline{k})^n$, defined over a field of degree $\leq \delta$, not in the torsion of the t-module, does there exist $c(\delta) = c_{a_0,\ldots,a_d,N,d,q,F}(\delta) > 0$ such that $\widehat{h}(\alpha) \geq c(\delta)$?; (ii) if (i) is satisfied, on a Drinfeld module of rank d, does there exist $c > 0$ such that, for any α not belonging to the torsion,

$$\widehat{h}(\alpha) \geq \frac{c}{\delta}?$$

The second problem is the extension of the classical Lehmer problem [Pao]. Denis partially solved Lehmer's problem ([Dis] Theorem 2) in the case of Carlitz modules, i.e. with $N = 1$ and $d = 1$ for which $\Phi(T)(x) = Tx + x^q$. He obtained the following minoration which is an analogue of Laurent's Theorem 2.4 for CM elliptic curves (elliptic Lehmer problem) and Dobrowolski's inequality (1.14):

Theorem 2.20 (Denis). *There exists a real number $\eta > 0$ such that, for any α belonging to the regular separable closure of k, not to the torsion, of degree $\leq \delta$, the minoration holds:*

$$\widehat{h}(\alpha) \ \geq \ \eta \, \frac{1}{\delta} \left(\frac{\mathrm{Log}\,\mathrm{Log}\,\delta}{\mathrm{Log}\,\delta} \right)^3$$

(the real number η is effective and computable from q).

Grandet-Hugot in [GtHt] studied analogues of Pisot and Salem numbers in fields of formal series: $x \in k_\infty$ is a Salem number if it is algebraic on k, $\deg(x) > 0$, and all its conjugates satisfy: $\deg(x_i) \leq 0$. In this context Denis ([Dis2] Theorem 1) proved the fact that there is no Salem number too close to 1, namely:

Theorem 2.21 (Denis). *Let $\alpha \in \overline{k}_\infty$ having at least one conjugate in k_∞. If α does not belong to the torsion, is of degree D on k, then*

$$\widehat{h}(\alpha) \ \geq \ \frac{1}{qD}$$

Extending the previous results, Denis ([Dis2] Theorem 3) solved Lehmer's problem for the following infinite family of t-modules:

Theorem 2.22 (Denis). *Let $\Phi(t) = a_0F^0 + a_1F + \ldots a_{d-1}F^{d-1} + F^d$ be a t-module of dimension 1 such that $a_i \in k_\infty \cap \overline{k}$, $0 \leq i \leq d-1$, is integral over $\mathbb{F}_q[T]$. Then there exists a real number $c_\Phi > 0$ depending only upon Φ, such that, if α is an algebraic element of k_∞, not in the torsion, of degree D on k, then*

$$\widehat{h}_\Phi(\alpha) \ \geq \ \frac{c_\Phi}{D}$$

The abelian Lehmer problem for Drinfeld modules was solved by David and Pacheco [DPao] using Denis's construction of the canonical height (with $A = \mathbb{F}_q[T]$):

Theorem 2.23 (David - Pacheco). *Let K/k be a finite extension, $\overline{K}$ an algebraic closure of K, and K^{ab} the largest abelian extension of K in $\overline{K}$. Let $\phi : A \to K\{\tau\}$ be a Drinfeld module of rank ≥ 1. Then there exists $c = c(\phi, K) > 0$ which depends only upon ϕ and K such that, for any $\alpha \in K^{\mathrm{ab}}$, not being in the torsion,*

$$\widehat{h}_\Phi(\alpha) \ \geq \ c.$$

In [Gca2] Ghioca investigates statements, for Drinfeld modules of generic characteristic, which would imply that the classical Lehmer problem for Drinfeld modules is true. In [Gca] Ghioca obtained several Lehmer type inequalities for the height of nontorsion points of Drinfeld modules. Using them, as consequence of Theorem 2.24 below, Ghioca proved several Mordell-Weil type structure theorems for Drinfeld modules over certain infinitely generated fields (the definitions of the terms can be found in [Gca]):

Theorem 2.24 (Ghioca). *Let $K/\mathbb{F}_q$ be a field extension, and $\phi : A \to K\{\tau\}$ be a Drinfeld module. Let L/K be a finite field extension. Let t be a non-constant element of A and assume that $\phi_t = \sum_{i=0}^{r} a_i \tau^i$ is monic. Let U be a good set of valuations on L and let $C(U)$ be the field of constants with respect to U. Let S be the finite set of valuations $v \in U$ such that ϕ has bad reduction at v. The degree of the valuation v is denoted by $d(v)$. Let $x \in L$.*

a) If S is empty, then either $x \in C(U)$ or there exists $v \in U$ such that $\widehat{h}_{U,v}(x) \geq d(v)$,

b) If S is not empty, then either $x \in \phi_{tors}$, or there exists $v \in U$ such that

$$\widehat{h}_{U,v}(x) > \frac{d(v)}{q^{2r+r^2N_\phi|S|}} \geq \frac{d(v)}{q^{r\left(2+(r^2+r)|S|\right)}}.$$

Moreover, if S is not empty and $x \in \phi_{tors}$, then there exists a polynomial $b(t) \in \mathbb{F}_q(t)$ of degree at most $rN_\phi|S|$ such that $\phi_{b(t)}(x) = 0$.

Let K be a finitely generated field extension of $\mathbb{F}_q$, and K^{alg} an algebraic closure of K. Ghioca [Gca2] developped global heights associated to a Drinfeld module $\phi : A \to K\{\tau\}$ and, for each divisor v, local heights $\widehat{h}_v : K^{alg} \to \mathbb{R}^+$ associated to ϕ. For Drinfeld modules of finite characteristic Ghioca [Gca2] obtained Lehmer type inequalities with the local heights, extending the classical Lehmer problem:

Theorem 2.25 (Ghioca). *For $\phi : A \to K\{\tau\}$ a Drinfeld module of finite characteristic, there exist two positive constants C and r depending only on ϕ such that if $x \in K^{alg}$ and v is a place of $K(x)$ for which $\widehat{h}_v(x) > 0$, then*

$$\widehat{h}_v(x) \geq \frac{C}{d^r}$$

where $d = [K(x) : K]$.

Bauchère [Bre] generalized David Pacheco's Theorem 2.23 to Drinfeld modules having complex multiplications, proving the abelian Lehmer problem in this context:

Theorem 2.26 (Bauchère). *Let ϕ be a A-Drinfeld module defined over $\overline{k}$ having complex multiplications. Let K/k be a finite field extension, L/K a Galois extension (finite or infinite) with Galois group G =Gal(L/K). Let H be a subgroup of the center of G and $E \subset L$ the subfield fixed by H. Let d_0 be an integer. We assume that there exists a finite place v of K such that $[E_w : K_v] \leq d_0$ for every place w of E, $v|w$. Then there exists a constant $c_0 = c_0(\phi) > 0$ such that, for any $\alpha \in L$, not belonging to the torsion for ϕ,*

$$\widehat{h}_\phi(\alpha) \geq \frac{1}{q^{c_0\, d(v)\, d_0^2\, [K:k]}}.$$

Theorem 2.26 is the analogue of a result obtained by Amoroso, David and Zannier [ADdZ] for the multiplicative group. Theorem 2.26 is particularly interesting when L/K is infinite. Bauchère [Bre] deduced special minorations of the heights $\widehat{h}_\phi(\alpha)$ in two Corollaries, for $L = K^{ab}$, and in the case where the subgroup H is trivial.

2.4 In other domains

The conjectural discontinuity of the Mahler measure $\mathrm{M}(\alpha), \alpha \in \overline{\mathbb{Q}}$, at 1 has consequences in different domains of mathematics. It is linked to the notions of "smallest complexity", "smallest growth rate", "smallest geometrical dilatation", "smallest geodesics", "smallest Salem number" or "smallest topological entropy" (Hironaka [Ha4]). We will keep an interdisciplinary viewpoint as in the recent survey [Sy6] by

C. Smyth and refer the reader to [GH] [Sy6]; we only mention below a few more or less new results. The smallest Mahler measures, or smallest Salem numbers, correspond to peculiar geometrical constructions in their respective domains.

2.4.1 Coxeter polynomials, graphs, Salem trees Let $\Gamma = (\Gamma_0, \Gamma_1)$ be a simple graph with set of enumerated vertices $\Gamma_0 = \{v_1, \ldots, v_n\}$, Γ_1 being the set of edges where $(v_i, v_j) \in \Gamma_1$ if there is an edge connecting the vertices v_i and v_j. The adjacency matrix of Γ is $\mathrm{Ad}_\Gamma := [a_{ij}] \in M_n(\mathbb{Z})$ where $a_{ij} = 1$, if $(v_i, v_j) \in \Gamma_1$ and $a_{ij} = 0$ otherwise. Assume that Γ is a tree. Denote by W_Γ the Weyl group of Γ, generated by the reflections $\sigma_1, \sigma_2, \ldots, \sigma_n$ and $\Phi_\Gamma := \sigma_1 \cdot \sigma_2 \cdot \ldots \cdot \sigma_n \in W_\Gamma$ the Coxeter transformation of Γ. The Coxeter polynomial of Γ is the characteristic polynomial of the Coxeter transformation $\Phi_\Gamma : \mathbb{R}^n \to \mathbb{R}^n$:

$$\mathrm{cox}_\Gamma(x) := \det(x \cdot \mathrm{Id}_n - \Phi_\Gamma) \in \mathbb{Z}[x].$$

Coxeter (1934) showed remarkable properties of the roots of the Coxeter polynomials. Coxeter polynomials were extensively studied for Γ any simply laced Dynkin diagram $\mathbb{A}_n, \mathbb{D}_n$ and $\mathbb{E}_n$. For $\Gamma = \mathbb{E}_n$, Gross, Hironaka and McMullen [GsHaMln] have obtained the factorization of Coxeter polynomials $\mathrm{cox}_\Gamma(x)$ as products of cyclotomic polynomials and irreducible Salem polynomials. In particular, $\mathrm{cox}_{\mathbb{E}_{10}}(x) := x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$ is Lehmer's polynomial. A tree $\mathscr{T}$ is said to be *cyclotomic*, resp. *a Salem tree*, if $\mathrm{cox}_{\mathscr{T}}(x)$ is a product of cyclotomic polynomials, resp. the product of cyclotomic polynomials by an irreducible Salem polynomial. Such objects generalize $\mathbb{E}_n$ as far as their Coxeter polynomials remains of the same form. Evripidou [Eiu], following Lakatos [Los] [Los2] [Los4] [Los5] and [GsHaMln], obtained structure theorems and formulations for the Coxeter polynomials of families of Salem trees, for the spectral radii of the respective Coxeter transformations. Lehmer's problem asks whether there exists a Salem tree of minimal Salem number; what would be its decomposition?

The Mahler measure $\mathrm{M}(G)$ of a finite graph G, with n vertices, is introduced in McKee and Smyth [MS]. If $D_G(z)$ is the characteristic polynomial of G, then the reciprocal integer polynomial associated with G is $z^n D_G(z + 1/z)$. The Mahler measure of this later polynomial is the Mahler measure $\mathrm{M}(G)$ of G; explicitely,

$$\mathrm{M}(G) = \prod_{D_G(\chi)=0, |\chi|>2} \frac{1}{2}(|\chi| + \sqrt{\chi^2 - 4}).$$

Cooley, McKee and Smyth [CyMS] [MS] [MS2] [MS3] studied Lehmer's problem from various constructions of finite graphs. They prove ([CyMS] Theorem 1 and Figures 1 to 3) that every connected non-bipartite graph that has Mahler measure smaller than the golden mean $1.618\ldots$ is one of the following type: (i) an odd cycle, (ii) a kite graph, (iii) a balloon graph, or (iv) one of the eight sporadic examples $Sp_a, \ldots, Sp_h$.

2.4.2 Growth series of groups, Coxeter groups, Coxeter systems Let G be an infinite group. Assume that G admits a finite generating set S. Define the length of an element g in $G = (G,S)$ to be the least nonnegative integer n such that g can be expressed as a product of n elements from $S \cup S^{-1}$. For every nonnegative integer n let $N_S(n)$ be the number of elements in G with length n. Following Milnor [Mor] the growth series of the group (G,S) is by definition

$$f(x) = \sum_{n=1}^{\infty} N_S(n)x^n, \qquad \text{for which } N_S(n) \leq (2|S|)^n.$$

The asymptotic *growth rate* of $G = (G,S)$, finite and ≥ 1, is by definition

$$\limsup_{n\to\infty} (N_S(n))^{1/n},$$

its inverse, positive, being the radius of convergence of $f(x)$. A Coxeter group G, with S being a finite generating set for G, is a group where every element of S has order two and all the other defining relators for G are of the form $(gh)^{m(g,h)} = 1_G$ where $m(g,h) = m(h,g)$ and $m(g,h) \geq 2$. Steinberg [Stg] and Bourbaki [Bki] showed that the growth series of a Coxeter group is a rational function. Salem numbers, Pisot numbers and Perron numbers occur as roots of the polynomials at the denominator (here the definition of a Salem number is often extended to quadratic Pisot numbers, conveniently and abusively).

Let us consider a hyperbolic cocompact Coxeter group G with generating set of reflections S acting in low dimensions $n \geq 2$.

Case $n = 2$: for the Coxeter reflection groups $G_{p_1,\ldots,p_d}$, with p_i any positive integer, of presentation $G_{p_1,\ldots,p_d} := (g_1,\ldots,g_d \mid (g_i)^2 = 1, (g_i g_{i+1})^{p_i} = 1)$, the denominator $\Delta_{p_1,\ldots,p_d}(x)$ of the growth series $f(x)$ of $G_{p_1,\ldots,p_d}$ is explicitely given by the following theorem [Cn].

Theorem 2.27 (Cannon-Wagreich [CnW], Floyd-Plotnick [FdPk], Parry [Paw]).

$$\Delta_{p_1,\ldots,p_d}(x) = [p_1][p_2]\ldots[p_d](x-d+1) + \sum_{i=1}^{d} [p_1]\ldots\widehat{[p_i]}\ldots[p_d]$$

The polynomial $\Delta_{p_1,\ldots,p_d}(x)$ is either a product of cyclotomic polynomials or a product of cyclotomic polynomials times an irreducible Salem polynomial. The Salem polynomial occurs if and only if $G_{p_1,\ldots,p_d}$ is hyperbolic, that is,

$$\frac{1}{p_1} + \ldots + \frac{1}{p_d} \; < \; d-2.$$

Then hyperbolic Coxeter reflection groups have Salem numbers as asymptotic growth rates. Such Salem numbers form a subclass of the set of Salem numbers. Lehmer's polynomial is $\Delta_{2,3,7}(x)$, denominator of the growth series of the (2,3,7)-hyperbolic triangle group (Takeuchi [Thi]). The *Construction of Salem* [Sa2] [Bo2], for establishing the existence of sequences of Salem numbers converging to a given

Pisot number, on the left and on the right, admits an analogue in terms of geometric convergence for the fundamental domains of cocompact planar hyperbolic Coxeter groups. Using the Construction of Salem Parry [Paw] gives a new proof of Theorem 2.27. Bartholdi and Ceccherini-Silberstein [BCSn] studied the Salem numbers which arise from some hyperbolic graphs. Hironaka [GH] solves the problem of Lehmer for the subclass of Salem numbers occuring as such asymptotic growth rates:

Theorem 2.28 (Hironaka [Ha]). *Lehmer's number is the smallest Salem number occuring as dominant roots of* $\Delta_{p_1,\ldots,p_d}$ *polynomials for any* $(p_1,\ldots,p_d)$*,* p_i *being positive integers.*

Case $n=3$: Parry [Paw] extends its 2-dimensional approach to every hyperbolic cocompact reflection Coxeter group on $\mathbb{H}^3$ generated by reflections whose fundamental domain is a bounded polyhedron (not just tetrahedron). Parry's approach is based on the properties of anti-reciprocal rational functions with Salem numbers. Kolpakov [Kpv] provides a generalization to the three-dimensional case, by establishing a metric convergence of fundamental domains for cocompact hyperbolic Coxeter groups with finite-volume limiting polyhedron; for instance, the compact polyhedra $\mathscr{P}(n) \subset \mathbb{H}^3$ of type $< 2,2,n,2,2 >$ converging, as $n \to \infty$, to a polyhedron $\mathscr{P}_\infty$ with a single four-valent ideal vertex. In this context, Kolpakov investigates the growth series of Coxeter groups acting on $\mathbb{H}^n$, $n \geq 3$ and their limit properties. The growth rates of ideal Coxeter polyhedra in $\mathbb{H}^3$ was studied by Nonaka [Nka].

Case $n \geq 4$: the growth rates of cocompact hyperbolic Coxeter groups are not Salem numbers anymore. Kellerhals and Perren [KlsP], §3 Example 2, show this fact with the example of the compact right-angled 120-cell in $\mathbb{H}^4$.

Lehmer's problem asks about the geometry of the poles of the growth rates of hyperbolic Coxeter groups acting on $\mathbb{H}^n$, and structure theorems about such groups having denominators of growth series of minimal Mahler measure.

Conjecture 14. *(Kellerhals-Perren) Let G be a Coxeter group acting cocompactly on* $\mathbb{H}^n$ *with natural generating set S and growth series* $f_S(x)$*. Then,*

(a) for n even, $f_S(x)$ *has precisely* $\frac{n}{2}$ *poles* $0 < x_1 < \ldots < x_{\frac{n}{2}} < 1$ *in the open unit interval* $(0,1)$ *;*

(b) for n odd, $f_S(x)$ *has precisely the pole 1 and* $\frac{n-1}{2}$ *poles* $0 < x_1 < \ldots < x_{\frac{n-1}{2}} < 1$ *in the interval* $(0,1)$*.*

In both cases, the poles are simple, and the non-real poles of $f_S(x)$ *are contained in the annulus of radii* x_m *and* x_m^{-1} *for some* $m \in \{1,\ldots,\lfloor \frac{n}{2} \rfloor\}$*.*

Theorem 2.29 (Kellerhals-Perren). *Let G be a Lannér group, an Esselmann group or a Kaplinskaya group, respectively, acting with natural generating set S on* $\mathbb{H}^4$*. Then,*

(1) the growth series $f_S(x)$ *of G is a quotient of relatively prime, monic and reciprocal polynomials of equal degree over the integers,*

(2) the growth series $f_S(x)$ of G possesses four distinct positive real poles appearing in pairs (x_1, x_1^{-1}) and (x_2, x_2^{-1}) with $x_1 < x_2 < 1 < x_2^{-1} < x_1^{-1}$; these poles are simple,

(3) the growth rate $\tau = x_1^{-1}$ is a Perron number,

(4) the non-real poles of $f_S(x)$ are contained in an annulus of radii x_2, x_2^{-1} around the unit circle,

(5) the growth series $f_S(x)$ of the Kaplinskaya group G_{66} with graph K_{66} has four distinct negative and four distinct positive simple real poles; for $G \neq G_{66}$, $f_S(x)$ has no negative pole.

Kellerhals and Kolpakov [KlsKv] (2014) prove that the simplex group $(3,5,3)$ has the smallest growth rate among all cocompact hyperbolic Coxeter groups on $\mathbb{H}^3$, and that it is, as such, unique. The growth rate is the Salem number $\tau' = 1.35098\ldots$ of minimal polynomial $X^{10} - X^9 - X^6 + X^5 - X^4 - X + 1$. Their approach provides a different proof for the analog situation in $\mathbb{H}^2$ where Hironaka [Ha] identified Lehmer's number as the minimal growth rate among all cocompact planar hyperbolic Coxeter groups and showed that it is (uniquely) achieved by the Coxeter triangle group $(3,7)$.

After Meyerhoff who proved that among all cusped hyperbolic 3- orbifolds the quotient of $\mathbb{H}^3$ by the tetrahedral Coxeter group $(3,3,6)$ has minimal volume, Kellerhals [Kls] (2013) proves that the group $(3,3,6)$ has smallest growth rate among all non- cocompact cofinite hyperbolic Coxeter groups, and that it is as such unique. This result extends to three dimensions some work of Floyd [Fd] who showed that the Coxeter triangle group $(3,\infty)$ has minimal growth rate among all non-cocompact cofinite planar hyperbolic Coxeter groups. In contrast to Floyd's result, the growth rate of the tetrahedral group $(3,3,6)$ is not a Pisot number. The following Theorem completes the picture of growth rate minimality for cofinite hyperbolic Coxeter groups in three dimensions.

Theorem 2.30 (Kellerhals). *Among all hyperbolic Coxeter groups with non-compact fundamental polyhedron of finite volume in $\mathbb{H}^3$, the tetrahedral group $(3,3,6)$ has minimal growth rate, and as such the group is unique.*

In [KU] Komori and Umemoto, for three-dimensional non-compact hyperbolic Coxeter groups of finite covolume, show that the growth rate of a three-dimensional generalized simplex reflection group is a Perron number. In [KY] Komori and Yukita show that the growth rates of ideal Coxeter groups in hyperbolic 3-space are also Perron numbers ; a Coxeter polytope P is called ideal if all vertices of P are located on the ideal boundary of hyperbolic space. They prove that the growth rate τ of an ideal Coxeter polytope with n facets in $\mathbb{H}^n$ satisfies $n-3 \leq \tau \leq n-1$. The smallest growth rates occur only if $n \leq 4$. They prove that the minimum of the growth rates is $0.492432^{-1} \approx 2.03074$, which is uniquely realized by the ideal Coxeter simplex with p = q = s = 2. In [U] Umemoto shows that infinitely many 2-Salem numbers can be realized as the growth rates of cocompact Coxeter groups in $\mathbb{H}^4$; the Coxeter

polytopes are here constructed by successive gluing of Coxeter polytopes, which are called Coxeter dominoes [U2]. The algebraic integers having a fixed number of conjugates outside the closed unit disk were studied by Bertin [Bn], Kerada [Kda], Samet [Set], Zaimi [Zi] [Zi2], in particular 2-Salem numbers in [Kda] to which Umemoto refers. In [ZZ] Zehrt and Zehrt-Liebendörfer construct infinitely many growth series of cocompact hyperbolic Coxeter groups in $\mathbb{H}^4$, whose denominator polynomials have the same distribution of roots as 2-Salem polynomials; their Coxeter polytopes are the Coxeter garlands built by the compact truncated Coxeter simplex described by the Coxeter graph on the left side of Figure 1 in [ZZ]. Lehmer's problem asks about the minimality of the houses of the 2-Salem numbers involved in these constructions. In [KlsN] Kellerhals and Nonaka prove that the growth rates of three-dimensional Coxeter groups (Γ, S) given by ideal Coxeter polyhedra of finite volume are Perron numbers.

A Coxeter system (W,S) is a Coxeter group W with a finite generating set S; the permuted products $s_{\sigma(1)}s_{\sigma(2)}\dots s_{\sigma(n)}$, $\sigma \in S_n$, are the *Coxeter elements* of (W,S). The element $w \in W$ is said to be *essential* if it is not conjugate into any subgroup $W_I \subset W$ generated by a proper subset $I \subset S$. The Coxeter group (W,S) acts naturally by reflections on $V \equiv \mathbb{R}^S$. Let $\lambda(w)$ be the spectral radius of $w|V$. When $\lambda(w) > 1$ it is also an eigenvalue of w. MacMullen [Mln4] proves the three following results.

Theorem 2.31 (MacMullen). *Let (W,S) be a Coxeter system and let $w \in W$ be essential. Then*

$$\lambda(w) \geq \inf_{S_n} \lambda(s_{\sigma(1)}s_{\sigma(2)}\dots s_{\sigma(n)}).$$

Let $\alpha(W,S)$ be the dominant eigenvalue of the adjacency matrix (A_{ij}) of (W,S), defined by $A_{ij} = 2\cos(\pi/m_{ij})$ for $i \neq j$ and $A_{ii} = 0$. Let $\beta(W,S)$ be the largest root of the equation $\beta + \beta^{-1} + 2 = \alpha(W,S)^2$ provided $\alpha(W,S) \geq 2$. If $\alpha(W,S) < 2$ then we put: $\beta(W,S) = 1$. Then $\lambda(w) = \beta(W,S)$ for all bicolored Coxeter element.

Theorem 2.32 (McMullen). *For any Coxeter system (W,S), we have*

$$\inf_{S_n} \lambda(s_{\sigma(1)}s_{\sigma(2)}\dots s_{\sigma(n)}) \geq \beta(W,S).$$

Theorem 2.33 (McMullen). *There are 38 minimal hyperbolic Coxeter systems (W,S), and among these the infimum* $\inf \beta(W,S)$ *is Lehmer's number.*

Lehmer's problem is solved in this context. The quantity $\beta(W,S)$ can be viewed as a measure (not in logarithmic terms) of the complexity of a Coxeter system. Denote by $Y_{a,b,c}$ the Coxeter system whose diagram is a tree with 3 branches of lenghts a,b and c, joined by a single node. MacMullen [Mln4] shows that the smallest Salem numbers of respective degrees 6, 8 and 10 coincide with $\lambda(w)$ for the Coxeter elements of $Y_{3,3,4}$, $Y_{2,4,5}$ and $Y_{2,3,7}$ respectively; in particular Lehmer's number is $\lambda(w)$

for the Coxeter elements of $Y_{2,3,7}$. MacMullen shows that the set of irreducible Coxeter systems with $\beta(W,S) < \Theta$ consists exactly of $Y_{2,4,5}$ and $Y_{2,3,n}$, $n \geq 7$. He shows that the infimum of $\beta(W,S)$ over all high-rank Coxeter systems coincides with Θ. There are 6 Salem numbers < 1.3 that arise as eigenvalues in Coxeter groups, five of them arising from the Coxeter elements of $Y_{2,3,n}$, $7 \leq n \leq 11$.

2.4.3 Mapping classes: small stretch factors We refer to Birman [Bin], Farb and Margalit [FbMt] and Hironaka [Ha6]. If S is a surface the *mapping class group* of S, denoted by $\mathrm{Mod}(S)$, is the group of isotopy classes of orientation-preserving diffeomorphisms of S (that restrict to the identity on ∂S if $\partial S \neq \emptyset$). An irreducible mapping class is an isotopy class of homeomorphisms f of a compact oriented surface S to itself so that no power preserves a nontrivial subsurface. The classification of Nielsen-Thuston states that a mapping class $[f] \in \mathrm{Mod}(S)$ is either periodic, reducible or pseudo-Anosov [FbMt] [FiLhPu]. In the periodic case, the situation is "analogous to roots of unity" in Lehmer's problem. The minoration problem of the Mahler measure finds its analogue in the minoration of the dilatation factors of the pseudo-Anosovs. We refer to a mapping class $[f]$ by one of its representive f.

Let S_g be a closed, orientable surface of genus $g \geq 2$ and $\mathrm{Mod}(S_g)$ its mapping class group. For any Pseudo-Anosov element $f \in \mathrm{Mod}(S_g)$, and any integer $0 \leq k \leq 2g$, let

(i) $\kappa(f)$ be the dimension of the subspace of $H_1(S_g, \mathbb{R})$ fixed by f (for which $0 \leq \kappa(f) \leq 2g$),

(ii) $h(f) = \mathrm{Log}\,(\lambda(f))$ be the entropy of f, as logarithm of the *stretch factor* $\lambda(f) > 1$ (or *dilatation*; the dilatation measures the dynamical complexity),

(iii)

$$L(k,g) := \min\{h(f) \mid f : S_g \to S_g \text{ and } \kappa(f) \geq k\}.$$

Thurston [FiLhPu] [Tn] noticed that the set of stretch factors for pseudo-Anosov elements of $\mathrm{Mod}(S_g)$ is closed and discrete in $\mathbb{R}$, and proved that any dilatation factor $\lambda(f) > 1$ is a Perron number, with $\lambda(f) + \lambda(f)^{-1}$ an algebraic integer of degree $\leq 4g-3$. The Perron number $\lambda(f)$ is the growth rate of lengths of curves under iteration (of any representant) of f, in any metric on S_g. These stretch factors appear as the length spectrum of the moduli space of genus g Riemann surface.

Penner [Per] proved that the asymptotic behaviour $L(0,g) \asymp 1/g$ holds. With $k = 2g$, Farb, Leininger and Margalit [FbLrMt] proved $L(2g,g) \asymp 1$. For the other values of k, since $L(0,g) \leq L(k,g) \leq L(2g,g)$, the following inequalities hold, from [AarDd] [KinTa] [Ha] [Per],

$$\frac{\mathrm{Log}\,2}{6}\left(\frac{1}{2g-2}\right) \leq L(k,g) \leq \mathrm{Log}\,(62).$$

For $k = 0$ and $g = 1$, $L(0,1) = \mathrm{Log}\left(\frac{3+\sqrt{5}}{2}\right)$ for $\mathbb{T}^2$. For $k = 0$ and $g = 2$, Cho and Ham [CoHm] [LuTt] [Zhv] proved $L(0,2) \approx 0.5435\ldots$, as logarithm of the largest root of the Salem polynomial $X^4 - X^3 - X^2 - X + 1$; these authors showed that this

minimum dilatation is given by Zhirov in [Zhv], and realized by Pseudo-Anosov 5-braids in [HmSg]. In [AlLrMt] Agol, Leininger and Margalit improved the upper bound to: $(2g-2)L(0,g) < \operatorname{Log}(\theta_2^{-4})$ for all $g \geq 2$, where θ_2^{-1} is the golden mean, and proved the main theorem:

$$L(k,g) \asymp \frac{k+1}{g}, \qquad g \geq 2, \quad 0 \leq k \leq 2g.$$

Arnoux-Yoccoz's Theorem [AxYz] states that, for $g \geq 2$, for any $C \geq 1$, there are only finitely many conjugacy classes in $\mathrm{Mod}(S_g)$ of pseudo-Anosov mapping classes with stretch factors at most C.

Minimal dilatation problem: what are the values of $L(k,g)$, except $L(0,g)$ for $g = 1,2$ already determined? i.e. what are the minima $\delta_g := \exp(L(0,g))$, $g \geq 3$?

Lower bounds of the entropy are difficult to establish: e.g. Penner [Per], Tsai [Tsi] on punctured surfaces, Boissy and Lanneau [ByLu] on translation surfaces that belong to a hyperelliptic component, Hironaka and Kin [HaKn]. Then Kin [Kin] [KinTa] [KinKjTa] formulated several questions and conjectures on the minimal dilatation problem and its realizations. Bauer [Ber] studied upper bounds of the least dilatations, and Minakawa [Mka] gave examples of pseudo-Anosovs with small dilatations. Farb, Leininger and Margalit [FbLrMt2] obtained a universal finiteness theorem for the set of all small dilatation pseudo-Anosov homeomorphisms $\phi : S \to S$, ranging over all surfaces S. The following questions were posed by in [Mln2] and [Fb].

Asymptotic behaviour: (i) Does $\lim_{g\to\infty} g\, L(0,g)$ exist? What is its value? (ii) Is the sequence $\{\delta_g\}_{g\geq 2}$ (strictly) monotone decreasing?

Kin, Kojima and Takasawa [KinKjTa], for monodromies of fibrations on manifolds obtained from the magic 3-manifold N by Dehn filling three cusps with some restriction, proved $\lim_{g\to\infty} g\, L(0,g) = \operatorname{Log}\left(\frac{3+\sqrt{5}}{2}\right)$; they also formulated limit conjectures for the asymptotic behaviour relative to compact surfaces of genus g with n boundary components.

A pseudo-Anosov mapping class $[f]$ is said to be *orientable* if the invariant (un)-stable foliation of a pseudo-Anosov homeomorphism $f \in [f]$ is orientable. Let $\lambda_H(f)$ be the spectral radius of the action of f on $H_1(S_g, \mathbb{R})$. It is the *homological stretch factor* of f. The inequality $\lambda_H(f) \leq \lambda(f)$ holds and equality occurs iff the invariant foliations for f are orientable. Stretch factors obey some constraints [Shin]: (i) $\deg(\lambda(f)) \geq 2$, (ii) $\deg(\lambda(f)) \leq 6g-6$, (iii) if $\deg(\lambda(f)) > 3g-3$, then $\deg(\lambda(f))$ is even. Shin [Shin] formulates the following questions.

Algebraicity of stretch factors: (i) Which real numbers can be the stretch factors, the homological stretch factors? (ii) What degrees of stretch factors can occur on S_g?

Let us define a mapping class $f_{g,k}$ by

$$f_{g,k} = (T_{c_g})^k \left(T_{d_g} \dots T_{c_2} T_{d_2} T_{c_1} T_{d_1}\right) \in \mathrm{Mod}(S_g),$$

where c_i and d_i are simple closed curves on S_g as in Figure 1 in [Shin], and T_c the Dehn twist about c.

Theorem 2.34 (Shin). *For each $g \geq 2, k \geq 3$, $f_{g,k}$ is a pseudo-Anosov mapping class which satisfies: (i) $\lambda(f_{g,k}) = \lambda_H(f_{g,k})$, (ii) $f_{g,k}$ is a Salem number, (iii) $\lim_{g\to\infty} f_{g,k} = k-1$, where the minimal polynomial of $\lambda(f_{g,k})$ is the irreducible Salem polynomial*

$$t^{2g} - (k-2)\Big(\sum_{j=1}^{2g-1} t^j\Big) + 1, \qquad \textit{of degree } 2g.$$

Shin [Shin] deduces that, for each $1 \leq h \leq g/2$, there exists a pseudo-Anosov mapping class $\widetilde{f_h} \in \mathrm{Mod}(S_g)$ such that $\deg(\widetilde{f_h}) = 2h$, with $\lambda(\widetilde{f_h})$ a Salem number. He conjectures that, on S_g, there exists a pseudo-Anosov mapping class with a stretch factor of degree d for any even $1 \leq d \leq 6g-6$. He proves that the conjecture is true for $g = 2$ to 5. Shin and Strenner [ShinSr] prove that the Perron numbers which are the stretch factors of pseudo-Anosov mapping classes arising from Penner's construction [Per] have conjugates which do not belong to the unit circle. In §3 in [ShinSr] they ask several questions about the geometry of the Galois conjugates of stretch factors, around the unit circle, obtained by several constructions: by Hironaka [Ha5], by Dunfeld and Tiozzo, by Lanneau and Thiffeault [LuTt] [LuTt2], by Shin [Shin]. For $S_{g,n}$ being an orientable surface with genus g and n marked points, Tsai [Tsi] proves that the infimum of stretch factors is of the order of $(\mathrm{Log}\, n)/n$ for $g \geq 2$ whereas it is of the order of $1/n$ for $g = 0$ and $g = 1$; Tsai asks the question about the asymptotic behaviour of this infimum of dilatation factors in the (g,n)-plane. For some subcollections of mapping classes, by generalizing Penner's construction and by comparing the smallness of dilatation factors with trivial homological dilatation, Hironaka [Ha8] concludes to the existence of subfamilies of pseudoanosovs which have asymptotically small dilatation factors.

In the context of $\mathbb{Z}^n$-actions on compact abelian groups (Proposition 17.2 and Theorem 18.1 in Schmidt [Sdt2]) the topological entropy is equal to the logarithm of the Mahler measure. If we assume that the stretch factors are Mahler measures $\mathrm{M}(\alpha)$ of algebraic numbers α (which are Perron numbers by Adler and Marcus [AM]), then we arrive at a contradiction since Penner [Per] showed that $L(0,g) \asymp \frac{1}{g}$ for surfaces of genus g. Indeed, it suffices to increase the genus g to find pseudo-Anosov elements of $\mathrm{Mod}(S_g)$ with dilatation factors arbitrarily close to 1, while Theorem 1.2 states that a discontinuity should exist. As a consequence of [Per], [Tsi] and of Theorem 1.2 (ex-Lehmer Conjecture) we deduce the following claims:

Assertion 1: *The stretch factors of the pseudo-Anosov elements of* $\mathrm{Mod}(S_g)$ *are Perron numbers which are not Mahler measures of algebraic numbers as soon as g is large enough.*

Assertion 2: *The stretch factors of the pseudo-Anosov elements of* $\mathrm{Mod}(S_{g,n})$*, where $S_{g,n}$ is an orientable surface with fixed genus g and n marked points, are Perron*

numbers which are not Mahler measures of algebraic numbers as soon as n is large enough.

Let S be a connected finite type oriented surface. Leininger [Lgr] considers subgroups of $\mathrm{Mod}(S)$ generated by two positive multi-twists; a multi-twist is a product of positive Dehn twists about disjoint essential simple closed curves. Given A and B two isotopy classes of essential 1-manifolds on S, we denote by T_A, resp. T_B, the positive multi-twist which is the product of positive Dehn twists about the components of A, resp. of B.

Theorem 2.35 (Leininger). *Any pseudo-Anosov element $f \in \langle T_A, T_B \rangle$ has a stretch factor which satisfies:*

$$\lambda(f) \geq \lambda_L \text{ (Lehmer's number).}$$

The realization occurs when S has genus 5 (with at most one marked point), $A = A_{Lehmer}, B = B_{Lehmer}$ given by Figure 1 in [Lgr] up to homeomorphism, and f conjugate to $(T_A T_B)^{\pm 1}$. Leininger's Theorem 2.35 is strikingly reminiscent of McMullen's Theorem 2.33. The following questions are formulated in §9.1 in [Lgr]:

Q1: Which Salem numbers occur as dilatation factors of pseudo-Anosov automorphisms?

Q2: Is there some topological condition on a pseudo-Anosov which guarantees that its dilatation factor is a Salem number?

Q3 (Lehmer's problem for dilatation factors): Is there an $\varepsilon > 1$ such that if f is a pseudo-Anosov automorphism in a finite co-area Teichmüller disk stabilizer, then $\lambda(f) \geq \varepsilon$?

Since dilatations factors of pseudo-Anosovs are Perron numbers and not necessarily Mahler measures of algebraic numbers (cf Assertions 1 and 2 above), Leininger's Theorem 2.35 and McMullen's Theorem 2.33 are addressed to the set of Salem numbers and suggest that Lehmer's number is actually the smallest Salem number in this set; meaning first that Lehmer's Conjecture is true for Salem numbers.

Let

$$f_{k,l}(t) := t^{2k} - t^{k+l} - t^k - t^{k-l} + 1,$$

$$\text{resp.} \qquad f_{x,y,z}(t) := t^{x+y-z} - t^x - t^y - t^{x-z} - t^{y-z} + 1,$$

and denote $\lambda_{(k,l)} > 1$, resp. $\lambda_{(x,y,z)} > 1$, its dominant root.

Related to the minimization problem is the one for orientable pseudo-Anosovs. The minimal dilatation factor for orientable pseudo-Anosov elements of $\mathrm{Mod}(S_g)$ is denoted by δ_g^+. The minimal dilatation problem for δ_g^+ is open in general. For $g = 2$, Zhirov [Zhv] obtained $\delta_2^+ = \delta_2$. For $g = 1$, $\delta_1^+ = \delta_1$ holds. From [Ha5] [LuTt], $\delta_g < \delta_g^+$ for $g = 4, 6, 8$. Hironaka [Ha5] obtained: (i) $\delta_g \leq \lambda_{(g+1,3)}$ if $g \equiv 0, 1, 3, 4 \pmod 6$ and $g \geq 3$, (ii) $\delta_g \leq \lambda_{(g+1,1)}$ if $g \equiv 2, 5 \pmod 6$ and $g \geq 5$, (iii) $\limsup_{g\to\infty} g \operatorname{Log} \delta_g \leq \operatorname{Log}\left(\frac{3+\sqrt{5}}{2}\right)$. Kin and Takasawa [KinTa3] complemented and improved these in-

equalities. They showed: (i) $\delta_g \leq \lambda_{(g+2,1)}$ if $g \equiv 0,1,5,6 \pmod{10}$ and $g \geq 5$, (ii) $\delta_g \leq \lambda_{(g+2,2)}$ if $g \equiv 7,9 \pmod{10}$ and $g \geq 7$; for $g \equiv 2,4 \pmod{10}$, under the assumption $g+2 \not\equiv 0 \pmod{4641}$, then they prove: (i) $\delta_g \leq \lambda_{(g+2,3)}$ if $\gcd(g+2,3) = 1$, (ii) $\delta_g \leq \lambda_{(g+2,7)}$ if $3|(g+2)$ and $\gcd(g+2,7) = 1$, (iii) $\delta_g \leq \lambda_{(g+2,13)}$ if $21|(g+2)$ and $\gcd(g+2,13) = 1$, (iv) $\delta_g \leq \lambda_{(g+2,17)}$ if $273|(g+2)$ and $\gcd(g+2,17) = 1$. For $g = 8$ and 13 they obtain the sharper upper bounds: $\delta_8 \leq \lambda_{(18,17,7)} < \lambda_{(9,1)}$ and $\delta_{13} \leq \lambda_{(27,21,8)} < \lambda_{(14,3)}$. For orientable pseudo-Anosovs, Lannneau and Thiffeault [LuTt]AIF obtained $\delta_3^+ = \lambda_{(3,1)}$, $\delta_4^+ = \lambda_{(4,1)}$, and the following lower bounds for $g = 6$ to 8: (i) $\delta_6^+ \geq \lambda_{(6,1)}$, (ii) $\delta_7^+ \geq \lambda_{(9,2)}$ and $\delta_8^+ \geq \lambda_{(8,1)}$. Hironaka [Ha5] gave the upper bounds: (i) $\delta_g^+ \leq \lambda_{(g+1,3)}$ if $g \equiv 1,3 \pmod 6$, (ii) $\delta_g^+ \leq \lambda_{(g,1)}$ if $g \equiv 2,4 \pmod 6$, (iii) $\delta_g^+ \leq \lambda_{(g+1,1)}$ if $g \equiv 5 \pmod 6$. He obtained the asymptotics: $\limsup_{g\to\infty, g\not\equiv 0 \pmod 6} g \operatorname{Log} \delta_g^+ \leq \operatorname{Log}\left(\frac{3+\sqrt{5}}{2}\right)$ and, from [LuTt], the equality: $\delta_8^+ = \lambda_{(8,1)}$. Kin and Takasawa [KinTa3] gave the better upper bounds: (i) $\delta_g^+ \leq \lambda_{(g+2,2)}$ if $g \equiv 7,9 \pmod{10}$ and $g \geq 7$, (ii) $\delta_g^+ \leq \lambda_{(g+2,4)}$ if $g \equiv 1,5 \pmod{10}$ and $g \geq 5$. Moreover they proved: $\delta_7^+ = \lambda_{(9,2)}$ (Aaber and Dunfeld [AarDd] obtained it independently) and $\delta_5 < \delta_5^+$.

The realization of the minimal dilatations is a basic question, with the uniqueness problem, considered by many authors: associated with least volume [AarDd] [KinKjTa], braids [CoHm] [HmSg] [HaKn] [KinTa] [KinTa2] [LuTt2], monodromies [FbLrMt] [KinKjTa] [KinTa3], Coxeter graphs and Coxeter elements [Lgr] [Ha5] [Shin], quotient families of mapping classes [Ha10], self-intersecting curves [Dwl], homology of mapping tori [AlLrMt]. There exists several constructions of small dilatation families, e.g. by Hironaka [Ha7] [Ha9], McMullen [Mln2], Dehornoy [Dy3] with Lorenz knots.

2.4.4 Knots, links, Alexander polynomials, homology growth, Jones polynomials, lenticularity of zeroes, lacunarity Constructions of Alexander polynomials of knots and links are given in [Kui] [Mui] [Ron] [Srt] [Tuv]. Silver and Williams in [SWs2] (reported in [Sy6] § 4.2 for an overview) investigate the Mahler measures of various Alexander polynomials of oriented links with d components in a homology 3-sphere; they obtain theorems on limits of Mahler measures and Mahler measures of derivatives of d-variate Mahler measures by performing $1/q$ surgery ($q \in \mathbb{N}$) on the dth component, allowing $q \to \infty$. In particular they consider the topological realizability of the small Mahler measures and limit Mahler measures on various examples.

For Pretzel links Hironaka ([Ha] [Ha2], [Ha4], [GH] p. 308) solves the minimization problem for the subclass of Salem numbers defined in Theorem 2.27 by

Theorem 2.36 (Hironaka [Ha]). *Let $p_1, \ldots, p_d$ positive integers. Then the Alexander polynomial of the $(p_1, \ldots, p_d, -1, \ldots, -1)$-pretzel link (Coxeter link), where the number of -1's is $d-2$, with respect to a suitable orientation of its components, is $\Delta_{p_1,\ldots,p_d}(-x)$.*

Lehmer's polynomial of the variable "$-x$" is the Alexander polynomial of the $(-2,3,7)$-pretzel knot and the $(-2,3,7)$-pretzel knot is equivalent to the $(2,3,7,-1)$-pretzel knot. Theorem 2.28 follows from Theorem 2.36. The Mahler measure of the $(2,3,7,-1)$- pretzel knot is the minimum of the set of Mahler measures of Alexander polynomials of (suitably oriented) $(p_1,\ldots,p_d,-1,\ldots,-1)$-pretzel links, over all d in $2\mathbb{N}+1$.

It is natural to find counterparts of Lehmer's problem in Topology where several polynomial invariants [FsWs] [FYHLMO] [Jos], [Sy5] § 14.6, were associated to knots, links and braids, for which the notions of convergence and "limit" can be defined (as in [ChrKn] [Dy2] [Ha3] [SWs]), in addition to Alexander polynomials. Indeed a theorem of Seifert [Srt] asserts that (i) any monic reciprocal integer polynomial $P(x)$, (ii) which satisfies $|P(1)|=1$, is the Alexander polynomial of (at least) one knot, and conversely; Burde [Bue] extended it to fibered knots (cf Hironaka [Ha3]). A Theorem of Kanenobu [Kbu] asserts that any reciprocal monic integer polynomial $P(x)$ is, up to multiples of $x-1$, the Alexander polynomial of a fibered link. Let us recall that infinitely many knots may possess the same polynomial invariants (Morton [Mon], Kanenobu [Kbu2]).

Periodic homology and exponential growth: the r-fold cyclic covering $X_r(K)$ of a knot $K\subset\mathbb{S}^3$ admits topological invariants, i.e. homology groups $H_1(X_r(K),\mathbb{Z})$, which are also invariants of the knot K. To K is associated a sequence of Alexander polynomials $(\Delta_i), i\geq 1$, in a single variable, such that $\Delta_{i+1}|\Delta_i$. Likewise, to an oriented link with d components is associated a sequence of Alexander polynomials in d variables. In both cases, the first Alexander polynomial of the sequence is usually called the Alexander polynomial of the knot K, resp. of the link. For a knot K Gordon [Grdn] proved that the first Alexander invariant $\lambda_1(t)=\Delta_1(t)/\Delta_2(t)$ satisfies the following equivalence :

$$\lambda_1(t)|(t^n-1) \quad\Longleftrightarrow\quad H_1(X_r(K),\mathbb{Z}) \;\cong\; H_1(X_{r+n}(K),\mathbb{Z}) \;\text{ for all }\; r. \tag{2.29}$$

The equivalence (2.29) is an analogue of Kronecker's Theorem. Gordon used the Pierce numbers of the Alexander polynomial of K, for which a linear recurrence is expected as in [Le] [ErEW]. Gordon obtained periods which are not prime powers and how to find all of them for knots of a given genus.

Theorem 2.37 (Gordon). *There exists a knot K of genus g for which $H_1(X_r(K),\mathbb{Z})$ has proper period n if and only if $n=1$, or $n=$ $\mathrm{lcm}\{m_i \mid i=1,2,\ldots,r\}$, where the m_i's are all distinct, each has at least two distinct prime factors, and $\sum_{i=1}^{r}\Phi(m_i)\leq 2g$.*

Departing from "Kronecker's Theorem" Gordon conjectured that when some zero of $\Delta_1(t)$ is not a root of unity, then the order of $H_1(X_r(K),\mathbb{Z})$ grows exponentially with r. This conjecture was proved by Riley [Rey], with p-adic methods, and González-Acuña and Short [GAS]. Both used the Gel'fond-Baker theory of linear forms in the logarithms of algebraic numbers. Silver and Williams [SWs] extended the conjecture of Gordon and its proof for knots, where the "finite order of $H_1(X_r(K),\mathbb{Z})$" is replaced

by the "order of the torsion subgroup of $H_1(X_r(K),\mathbb{Z})$", and for links in $\mathbb{S}^3$. They identified the torsion subgroups with the connected components of periodic points in a dynamical system of algebraic origin [Sdt2], connected the limit with the logarithmic Mahler measure (for any finite-index subgroup $\Lambda \subset \mathbb{Z}^d$, the number of such connected components is denoted by b_Λ and $\langle\Lambda\rangle := \{|v| \mid v \in \Lambda \setminus \{0\}\}$ is the norm of the smallest nonzero vector of Λ; cf [SWs] for the definitions):

Theorem 2.38 (Silver-Williams [SWs]). *Let $l = l_1 \cup \ldots \cup l_d$ be an oriented link of d components having nonzero Alexander polynomial Δ, in d variables. Then*

$$\limsup_{\langle\Lambda\rangle\to\infty} \frac{1}{|\mathbb{Z}^d/\Lambda|} \mathrm{Log}\, b_\Lambda = \mathrm{Log}\, \mathrm{M}(\Delta) \tag{2.30}$$

where "$\limsup$*" is replaced by "*$\lim$*" if $d = 1$.*

Let M be a finitely generated module over $\mathbb{Z}[\mathbb{Z}^n]$ and $\widehat{M}$ its (compact) Pontryagin dual. For any subgroup $\Lambda \subset \mathbb{Z}^n$ of finite index, let b_Λ be the number of connected components of the set of elements of $\widehat{M}$ fixed by actions of the elements of Λ. Le ([Let], Theorem 1) proved a conjecture of K. Schmidt [Sdt] on the growth of the number b_Λ; as a consequence Le generalized ([Let], Theorem 2) Silver Williams's Theorem 2.38 on the growth of the homology torsion of finite abelian covering of link complements, with the logarithmic Mahler measure of the first nonzero Alexander polynomial of the link. In each case, since the growth is expressed by the logarithmic Mahler measure of the (first nonzero) Alexander polynomial, Lehmer's problem amounts to establishing a universal minorant > 0 of the exponential base. For nonsplit links in $\mathbb{S}^3$, that is in the nonabelian covering case, Le [Let] generalized Theorem 2.38 using the L^2-torsion, i.e. the hyperbolic volume in the rhs part of (2.30) instead of the logarithmic Mahler measure of the 0th Alexander polynomial; in such a case the minimality of Mahler measures would find its origin in the minimality of hyperbolic volumes [Let2].

The growth of the homology torsion depends upon the (nonzero) logarithmic Mahler measure of the Alexander polynomial(s) of a knot or a link. Hence the geometry of zeroes of Alexander polynomials is important for the minoration of the homology growth [GH]. At this step, let us briefly mention the importance of other studies on the roots of Alexander polynomials: (i) monodromies and dynamics of surface homeomorphisms [HaLi] [Ron], (ii) knot groups: factorization and divisibility [Mui], (iii) knot groups: orderability (Perron Rolfsen), (iv) statistical models (Lin Wang).

Applying solenoidal dynamical systems theory to knot theory enabled Noguchi [Nhi] to prove that the dominant coefficient a_n of the Alexander polynomial $\Delta_K(t) = \sum_{i=0}^n a_i t^i, a_0 a_n \neq 0$, of a knot K, α_i being the zeroes (counted with multiplicities) of $\Delta_K(t)$, satisfies ($|\cdot|_p$ is the p-adic norm normalized by $|p|_p = 1/p$ on $\mathbb{Q}_p$):

$$\mathrm{Log}\,|a_n| = \sum_{p<\infty} \sum_{|\alpha_i|_p>1} \mathrm{Log}\,|\alpha_i|_p$$

He proved that the distribution of zeroes measures a "distance" of the Alexander module from being finitely generated as a $\mathbb{Z}$-module, and that the growth of order of the first homology of the r-fold cyclic covering $X_r(K)$ branched over K is related to the zeroes by

$$\lim_{r\to\infty, |H_1(\cdot)|\neq 0} \frac{\mathrm{Log}\,|H_1(X_r(K);\mathbb{Z})|}{r} = \sum_{p\leq\infty} \sum_{|\alpha_i|_p>1} \mathrm{Log}\,|\alpha_i|_p.$$

Therefore the leading coefficient of $\Delta_K(t)$ is closely related to the homology growth and the p-adic norms of the zeroes α_i.

A link, or a knot, is said to be alternating if it admits a diagram where (along every component) the strands are passed under-over. In 2002 Hoste (Hirasawa and Murasugi [HraMi]) stated the following conjecture: *let K be an alternating knot and $\Delta_K(t)$ its Alexander polynomial. If α is a zero of $\Delta_K(t)$, then* $\Re e(\alpha) > -1$.

Hoste's Conjecture is proved in some cases: cf [HaLi] [LyhMi] [Stw] [Stw2]. The problem of the geometry and the boundedness of zeroes of the (knot and link) Alexander polynomials is difficult and related to two other conjectures on the coefficients of these polynomials, namely the Fox's trapezoidal Conjecture and the Log-concavity Conjecture [KfPr] [Stw] [Stw2].

In his studies of Lorenz knots [Dy] [Dy2] [Dy3], Dehornoy obtained the following much more precise statement on the geometry of the zero locus (g is the smallest genus of a surface spanning the knot; the braid index b is the smallest number of strands of a braid whose closure is the knot):

Theorem 2.39 (Dehornoy [Dy2]). *Let K be a Lorenz knot. Let g denote its genus and b its braid index. Then the zeroes of the Alexander polynomial of K lie in the annulus*

$$\{z\in\mathbb{C} \mid (2g)^{-4/(b-1)} \leq |z| \leq (2g)^{4/(b-1)}\}. \tag{2.31}$$

The Alexander polynomial of a Lorenz knot reflects an intermediate step between signatures and genus [Dy2]. A certain proportion of zeroes lie on the unit circle and are controlled by the ω-signatures (Gambaudo and Ghys, cited in [Dy2]). The other zeroes lie within a certain distance from the unit circle and are controlled by the house of the Alexander polynomial, which is the modulus of the largest zero. The problem of the minimality of the house of this Alexander polynomial is reminiscent of the Schinzel-Zassenhaus Conjecture if it were expressed as a function of its degree. For Lorenz knots this house is interpreted as follows: it is the growth rate of the associated homological monodromy (for details, cf [Dy2] § 2). Figure 3.3 in [Dy2] shows two examples of Lorenz knots, with respective braid index and genus $(b,g) = (40,100)$ and $= (100,625)$; interestingly, the distribution of zeroes within the annulus (2.31) appears angularly fairly regular (in the sense of Bilu's Theorem [Bu]) but exhibit lenticuli of zeroes in the angular sector $\arg(z) \in [\pi-\pi/2, \pi+\pi/2]$. Such lenticuli do exist for integer polynomials of small Mahler measure, of the variable "$-x$", and are shown to be at the origin of the minoration of the Mahler measure in the problem of Lehmer in the present note. Though Dehornoy did not publish (yet) further on the

Mahler measures of the Alexander polynomials of Lorenz knots, in particular in the way (b,g) tends to infinity, it is very probable that such polynomials are good candidates for giving small Mahler measures together with a topological interpretation of the houses. The above examples suggest that the Alexander polynomials of Lorenz knots are not Salem polynomials, though no proof seems to exist.

Before Le [Let] [Let2], Boyd and Rodriguez-Villegas [Bo17] [Bo18] [BoRs] studied the connections between the Mahler measure of the *A-polynomial of a knot* and the hyperbolic volume of its complement. A-polynomials were introduced in hyperbolic geometry by Cooper et al [CCGLS] (are not Alexander polynomials, though "A" is used in homage to Alexander). The irreducible factors of A-polynomials have (logarithmic) Mahler measures which are shown to be finite sums of Bloch-Wigner dilogarithms [GlZ] [Za3] of algebraic numbers. The values of such dilogarithms are related to Chinburg's Conjecture. Several examples are taken by the authors to investigate Chinburg's Conjecture and its generalization refered to as Boyd's question (cf also Ray [Ry]). Chinburg's Conjecture [Bo18] is stated as follows: for each negative discriminant $-f$ there exists a polynomial $P = P_f \in \mathbb{Z}[x,y]$ and a nonzero rational number r_f such that $\mathrm{Log}\,\mathrm{M}(P) = r_f \frac{f\sqrt{f}}{4\pi} L(2,\chi_f)$. Boyd's question is stated as follows: for every number field F having a number of complex embeddings equal to 1 (i.e. $r_2 = 1$), does there exist a polynomial $P = P_F \in \mathbb{Z}[x,y]$ and a rational number r_F such that $\mathrm{Log}\,\mathrm{M}(P) = r_F Z_F$?, where ζ_F is the Dedeking zeta function of F and

$$Z_F = \frac{3\,|\mathrm{disc}(F)|^{3/2}\,\zeta_F(2)}{2^{2n-3}\,\pi^{2n-1}};$$

the starting point being (Smyth [Sy4]): for $f = 3$, $\mathrm{Log}\,\mathrm{M}(1+x+y) = \frac{3\sqrt{3}}{4\pi} L(2,\chi_3)$.

Jones polynomials of knots and links, lacunarity in coefficient vectors:

Let L be a hyperbolic link and, for $m \geq 1$, denote by L_m the link obtained from L by adding m full twists on n strands [Ron] [ChrKn]. By Thurston's hyperbolic Dehn surgery, the volume $\mathrm{Vol}(\mathbb{S}^3 \setminus L_m)$ converges to $\mathrm{Vol}(\mathbb{S}^3 \setminus (L \cup U))$, as m tends to infinity, where U is an unknot encircling n strands of L such that L_m is obtained from L by a $-1/m$ surgery on U. More generally, let $\underline{m} = (1, m_1, \ldots, m_s)$, for $s \geq 1$, and $L_{\underline{m}} := L_{m_1,\ldots,m_s}$ the multi-twisted link obtained from a link diagram L by a $-1/m_i$ surgery on an unknot U_i, for $i = 1, \ldots, s$. In the following theorem convergence of Mahler measures has to be taken in the sense of the Boyd Lawton's Theorem 2.1.

Theorem 2.40 (Champanerkar - Kofman). *(i) The Mahler measure* $\mathrm{M}(V_{L_m}(t))$ *of the Jones polynomial of* L_m *converges to the Mahler measure of a 2-variable polynomial, as m tends to infinity;*

(ii) the Mahler measure $\mathrm{M}(V_{L_{\underline{m}}}(t))$ *of the Jones polynomial of* $L_{\underline{m}}$ *converges to the Mahler measure of a* $(s+1)$*-variable polynomial, as* $\underline{m}$ *tends to infinity.*

In [ChrKn] Theorem 2.4, Champanerkar and Kofman [ChrKn] extended Theorem 2.40 to the convergence of the Mahler measures of colored Jones polynomials $J_N(L_m,t)$ and $J_N(L_{\underline{m}},t)$ for fixed N, as m, resp. $\underline{m}$, tends to infinity; here coloring

means by the N-dimensional irreducible representation of $SL_2(\mathbb{C})$ with the normalization of $J_2(L_m,t)$ as $J_2(L_m,t) = (t^{1/2}+t^{-1/2})V_{L_m}(t)$, resp. for $\underline{m}$. They proved that the limit $\lim_{m\to\infty} \mathrm{M}(J_N(L_m,t))$, resp. for $\underline{m}$, is the Mahler measure of a multivariate polynomial. What smallness of limit Mahler measures can be reached by this construction, and what are the corresponding geometrical realizations?

In [ChrKn] (Theorem 2.5 and Corollary 3.2) Champanerkar and Kofman obtain the following theorem, reminiscent of the moderate lacunarity of the Parry Upper function occuring at small Mahler measure (Theorem 4.4, Theorem 4.6 and §5.3) and the limit equidistribution of conjugates on the unit circle (Theorem 1.7), which occur concomitantly in the (classical) Lehmer problem:

Theorem 2.41 (Champanerkar - Kofman). *Let $N \geq 1$ be a fixed integer. With the above notations,*

(i) let $\{\gamma_{i,m}\}$ be the set of distinct roots of the Jones polynomial $J_N(L_m,t)$. Then $\liminf_{\kappa\to\infty} \#\{\gamma_{i,m} \mid m \leq \kappa\} = \infty$*, and for any $\varepsilon > 0$, there exists an integer q_ε such that the number of such roots satisfies*

$$\#\{\gamma_{i,m} \mid ||\gamma_{i,m}| - 1| \geq \varepsilon\} < q_\varepsilon,$$

(ii) for m sufficiently large, the coefficient vector of the Jones polynomial $J_N(L_m,t)$ has nonzero fixed blocks of integer digits separated by gaps (blocks of zeroes) whose length increases as m tends to infinity.

In addition to the relative limitation of the multiplicities of the roots, Theorem 2.41 means that, in the annulus $1-\varepsilon < |z|-1 < 1+\varepsilon$, the clustering of the roots occurs, up to q_ε of them (densification), and is associated with a moderate lacunarity ("gappiness" in the sense of [VG]) of the Jones polynomials which increases with m. This Theorem has been extended to other Jones polynomials by these authors [ChrKn] and followed previous experimental observations. From Theorem 2.40 and Theorem 2.41 it is likely that such Jones polynomials lead to very small multivariate Mahler measures, at least are good candidates.

Other families of Jones polynomials, their zeroes and their limit distributions, were investigated, for which interesting limit Mahler measures may be expected: e.g. Chang and Shrock [CgSk], Wu and Wang [WuWg], Jin and Zhang [JnZg] [JnZg2] [JnZg3], related to models in statistical physics. The moderate lacunarities occurring in the coefficient vectors of Jones polynomials were studied by Franks and Williams [FsWs] in the context of polynomial invariants associated with braids, knot and links which generalize Alexander polynomials and Jones polynomials [FsWs] [FYHLMO] [Jos] [Mui].

2.4.5 Arithmetic Hyperbolic Geometry Leininger's constructions in [Lgr] give the dilatation factors of pseudo-Anosovs as spectral radii of hyperbolic elements in some Fuchsian groups. The minimality of the Salem numbers as dilatation factors is

defined in a more general context (Neuman and Reid [NmRd], Maclachlan and Reid [MlK], Ghate and Hironaka [GH] p. 303).

Theorem 2.42 (Neuman-Reid). *The Salem numbers are precisely the spectral radii of hyperbolic elements of arithmetic Fuchsian groups derived from quaternion algebras.*

Arithmetic hyperbolic groups are arithmetic groups of isometries of hyperbolic n-space $\mathbb{H}^n$. Vinberg and Shvartsman [VS] p. 217 have defined the large subclass of the arithmetic hyperbolic groups of the simplest type, in terms of an admissible quadratic form over a totally real number field K. This subclass includes all arithmetic hyperbolic groups in even dimensions, infinitely many wide-commensurability classes of hyperbolic groups in all dimensions [Ml], and all noncocompact arithmetic hyperbolic groups in all dimensions. Isometries of $\mathbb{H}^n$ are either elliptic, parabolic or hyperbolic. An isometry $\gamma \in \mathbb{H}^n$ is hyperbolic if and only if there is a unique geodesic L in $\mathbb{H}^n$, called the *axis* of γ, along which γ acts as a translation by a positive distance $l(\gamma)$ called the *translation length* of γ.

The following theorems generalize previous results of Neumann and Reid [NmRd] in dimension 2 and 3 and show the important role played by the smallest Salem numbers:

Theorem 2.43. *(Emery, Ratcliffe, Tschantz [ERTz]) Let Γ be an arithmetic group of isometries of $\mathbb{H}^n$, $n \geq 2$, of the simplest type defined over a totally real algebraic number K. Let $\Gamma^{(2)}$ be the subgroup of Γ of finite index generated by the squares of the elements of Γ. Let γ be a hyperbolic element of Γ, and let $\lambda = e^{l(\gamma)}$. If n is even or $\gamma \in \Gamma^{(2)}$, then λ is a Salem number such that $K \subset \mathbb{Q}(\lambda + \lambda^{-1})$ and* $\deg_K(\lambda) \leq n+1$.

Conversely, if $\lambda \in T$, K is a subfield of $\mathbb{Q}(\lambda + \lambda^{-1})$ and n such that $\deg_K(\lambda) \leq n+1$*, then there exists an arithmetic group Γ of isometries of $\mathbb{H}^n$ of the simplest type defined over K and a hyperbolic element $\gamma \in \Gamma$ such that $\lambda = e^{l(\gamma)}$.*

Theorem 2.44. *(Emery, Ratcliffe, Tschantz [ERTz]) Let Γ be an arithmetic group of isometries of $\mathbb{H}^n$, $n \geq 2$ odd, of the simplest type defined over a totally real algebraic number K. Let $\Gamma^{(2)}$ be the subgroup of Γ of finite index generated by the squares of the elements of Γ. Let γ be a hyperbolic element of Γ, and let $\lambda = e^{l(\gamma)}$. Then λ is a Salem number which is square-rootable over K.*

Conversely, if $\lambda \in T$, K is a subfield of $\mathbb{Q}(\lambda + \lambda^{-1})$ and n an odd positive integer such that $\deg_K(\lambda) \leq n+1$*, and λ is square-rootable over K, then there exists an arithmetic group Γ of isometries of $\mathbb{H}^n$ of the simplest type defined over K and a hyperbolic element $\gamma \in \Gamma$ such that $\sqrt{\lambda} = e^{l(\gamma)}$.*

2.4.6 Salem numbers and Dynamics of Automorphisms of Complex Compact Surfaces Let X be a compact Kähler variety and f an automorphism of X. The automorphism f induces an invertible linear map f^* on $H^*(X,\mathbb{C})$, resp. $H^*(X,\mathbb{R})$, $H^*(X,\mathbb{Z})$, which preserves the Hodge decomposition, the intersection form, the Kähler

cone. Iterating f provides a dynamical system to which real algebraic integers ≥ 1 are associated. The greatest eigenvalue of the action of f on $H^*(X,\mathbb{C})$ is usually called the *maximal dynamical degree* of f. This terminology is the same as the one used for the β-shift in the present note, but the notions are different. The maximal dynamical degree of f is denoted by $\lambda(f)$; it is related to the topological entropy $h_{top}(f)$ of f by $\operatorname{Log}\lambda(f) = h_{top}(f)$ by a Theorem of Gromov and Yomdin [Gv][Yn]. Saying that an automorphism is of positive entropy is equivalent to saying that its maximal dynamical degree is > 1. In particular if X is a surface the characteristic polynomial of f^* on $H^2(X,\mathbb{Z})$ is a (not necessarily irreducible) Salem polynomial (McMullen [Mln3]); the maximal dynamical degree $\lambda(f)$ of f is the spectral radius of f^* on $H^{1,1}(X)$ and is a Salem number. Salem numbers are deeply linked to the geometry of the surface. Among all complex compact surfaces [BHPvV], Cantat [Ctt] [Ctt2] showed that, if X is a complex compact surface for which there exists an automorphism of X having a positive entropy, then there exists a birational morphism from X to a torus, a $K3$ surface, a surface of Enriques, or the projective plane. Therefore it suffices to consider complex tori (Oguiso and Truong [OgoTg], Reschke [Rke] 2017), Enriques surfaces (Oguiso [Ogo] [Ogo3]), and $K3$ surfaces (Gross and McMullen [GsMln], McMullen [Mln3], Oguiso [Ogo2], Shimada [Sda]) if X is not rational.

The restriction to compact Kähler surfaces is justified by the fact that the topological entropy of all automorphisms vanishes on compact complex surfaces which are not Kähler (Cantat [Ctt2]). The existence of an automorphism of positive entropy is a deep question [Bst] [Bst2] [CttD] [EOgoY] [Mln3] [Ogo4] [OgoTg2].

On each type of surface, what are the Salem numbers which appear? In this context the problem of Lehmer can be formulated by asking what are the minimal Salem numbers which occur, per type of surface, and the corresponding geometrical realizations.

In [Mln] McMullen gives a general construction of $K3$ surface automorphisms f from unramified Salem numbers, such that, for every such automorphism f, the topological entropy $\operatorname{Log}\lambda(f)$ is positive, together with a criterion for the resulting automorphism to have a Siegel disk (domains on which f acts by an irrational rotation). The Salem polynomials involved, of the respective dynamical degrees $\lambda(f)$, have degree 22, trace -1 and are associated to an even unimodular lattice of signature $(3,19)$ on which f acts as an isometry, by the Theorem of Torelli. The surface is non-projective to carry a Siegel disk.

McMullen [Mln5] (Theorem A.1) proved that Lehmer's number (denoted by λ_{10}) is the smallest Salem number that can appear as dynamical degree of an automorphism of a complex compact surface:

$$h(f) \geq \operatorname{Log}\lambda_{10} = 0.162357\ldots.$$

He gave a geometrical realization of Lehmer's number in [Mln5] on a rational surface (cf also Bedford and Kim [BdKm]), in [Mln6] on a nonprojective $K3$ surface, in [Mln7] on a projective $K3$ surface. On the contrary Oguiso [Ogo2] proved that Lehmer's number cannot be realized on an Enriques surface. In [Mln7] McMullen

proved that the value $\mathrm{Log}\,\lambda_d$ arises as the entropy of an automorphism of a complex projective $K3$ surface if

$$d = 2,4,6,8,10 \text{ or } 18, \text{but not if } d = 14,16 \text{ or } d \geq 20.$$

Brandhorst and González-Alonso [BstG] completed the above "realizability" list with the value $d = 12$ (Theorem 1.2 in [Mln7]).

For projective surfaces, the degree of the Salem number is bounded by the rank of the Néron-Severi group; for $K3$ surfaces in characteristic zero it is at most 20, due to Hodge theory. In positive characteristic the rank 22 is possible (case of supersingular $K3$ surfaces) [Bst2] [Yu]. Therefore all such Salem numbers, when less than 1.3, are listed in Mossinghoff's list in [Mlist], the list being complete up to degree 44.

Reschke [Rke] [Rke2] gave a necessary and sufficient condition for a Salem number to be realized as dynamical degree of an automorphism of a complex torus, with degrees 2, 4 or 6; moreover he investigated the relations between the values of the Salem numbers and the corresponding geometry and projectiveness of the tori. Zhao [Zhao] extended the method of Reschke for tori endowed with real structures, showing that it suffices to consider real abelian surfaces. Zhao classified such real abelian surfaces into 8 types according to the number of connected components and the simplicity of the underlying complex abelian surface. For each type the set of Salem numbers which can be realized by real automorphisms is determined. Zhao [Zhao] proved that Lehmer's number cannot be realized by a real $K3$ surface.

Dolgachev [Dgv] investigated automorphisms on Enriques surfaces of dynamical degrees > 1 which are small Salem numbers, of small degree 2 to 10 (Salem numbers of degree 2 are quadratic Pisot numbers). The method does not allow to conclude on the minimality of the Salem numbers. The author uses the lower semi-continuity properties of the dynamical degree of an automorphism g of an algebraic surface S when (S,g) varies in an algebraic family.

In positive characteristic Brandhorst and González-Alonso [BstG] proved that the values $\mathrm{Log}\,\lambda_d$ arise as the entropy of an automorphism of a supersingular $K3$ surface over a field of characteristic $p = 5$ if and only if $d \leq 22$ is even and $d \neq 18$, giving in their Appendix B the list of Salem numbers λ_d of degree d and respective minimal polynomials. They develop a strategy to characterize the minimal Salem polynomials, in particular their cyclotomic factors, for various realizations in supersingular $K3$ surfaces having Artin invariants σ ranging from 1 to 7, in characteristic 5. Yu [Yu] studied the maximal degrees of the Salem numbers arising from automorphisms of $K3$ surfaces, defined over an algebraically closed field of characteristic p, in terms of the elliptic fibrations having infinite automorphism groups, and Artin invariants.

Oguiso and Truong [OgoTg] Dinh, Nguyen and Truong [DhNn] [DhNnTg] investigated the structure of compact Kähler manifolds, in dimension ≥ 3, from the point of view of establishing relations between non-trivial invariant meromorphic fibrations, pseudo-automorphisms f and the dynamical degrees $\lambda_k(f)$. Lehmer's problem can be formulated by asking when the first dynamical degree $\lambda_1(f)$ is a Salem number,

what minimal value for $\lambda_1(f)$ can be reached and what are the possible geometrical realizations for the minimal ones.

3 Asymptotic expansions of the Mahler measures $\mathrm{M}(-1+X+X^n)$

3.1 Factorization of the trinomials $-1+X+X^n$, lenticuli of roots

The notations used throughout this note come from the factorization of $G_n(X) := -1+X+X^n$ (Selmer [Sr], Verger-Gaugry [VG6] Section 2). Summing in pairs over complex conjugated imaginary roots, the indexation of the roots and the factorization of $G_n(X)$ are taken as follows:

$$G_n(X) = (X-\theta_n)\left(\prod_{j=1}^{\lfloor \frac{n}{6} \rfloor}(X-z_{j,n})(X-\overline{z_{j,n}})\right) \times q_n(X), \tag{3.1}$$

where θ_n is the only (real) root of $G_n(X)$ in the interval $(0,1)$, where

$$q_n(X) = \begin{cases} \left(\prod_{j=1+\lfloor \frac{n}{6} \rfloor}^{\frac{n-2}{2}}(X-z_{j,n})(X-\overline{z_{j,n}})\right) \times (X-z_{\frac{n}{2},n}) & \text{if } n \text{ is even, with} \\ & z_{\frac{n}{2},n} \text{ real} < -1, \\ \prod_{j=1+\lfloor \frac{n}{6} \rfloor}^{\frac{n-1}{2}}(X-z_{j,n})(X-\overline{z_{j,n}}) & \text{if } n \text{ is odd,} \end{cases}$$

where the index $j = 1,2,\ldots$ is such that $z_{j,n}$ is a (nonreal) complex zero of $G_n(X)$, except if n is even and $j = n/2$, such that the argument $\arg(z_{j,n})$ of $z_{j,n}$ is roughly equal to $2\pi j/n$ (Proposition 3.7) and that the family of arguments $(\arg(z_{j,n}))_{1\le j<\lfloor n/2 \rfloor}$ forms a strictly increasing sequence with j:

$$0 < \arg(z_{1,n}) < \arg(z_{2,n}) < \ldots < \arg(z_{\lfloor \frac{n}{2} \rfloor,n}) \le \pi.$$

For $n \ge 2$ all the roots of $G_n(X)$ are simple, and the roots of $G_n^*(X) = 1+X^{n-1}-X^n$, as inverses of the roots of $G_n(X)$, are classified in the reversed order (Figure 1).

Proposition 3.1. *Let $n \ge 2$. If $n \not\equiv 5 \pmod 6$, then $G_n(X)$ is irreducible over $\mathbb{Q}$. If $n \equiv 5 \pmod 6$, then the polynomial $G_n(X)$ admits X^2-X+1 as irreducible factor in its factorization and $G_n(X)/(X^2-X+1)$ is irreducible.*

Proof. Selmer [Sr]. □

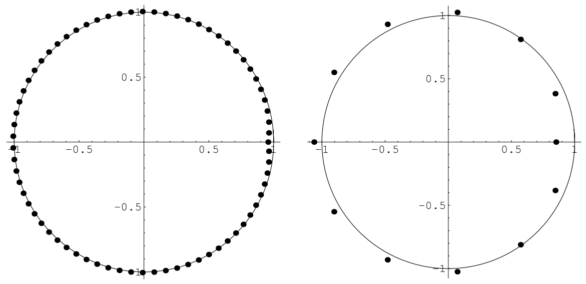

Figure 1. The roots (black bullets) of $G_n(z)$ (represented here with $n = 71$ and $n = 12$) are uniformly distributed near $|z| = 1$ according to the theory of Erdős-Turán-Amoroso-Mignotte. A slight bump appears in the half-plane $\Re(z) > 1/2$ in the neighbourhood of 1, at the origin of the different regimes of asymptotic expansions. The dominant root of $G_n^*(z)$ is the Perron number $\theta_n^{-1} > 1$, with θ_n the unique root of G_n in the interval $(0,1)$.

Proposition 3.2. *For all $n \geq 2$, all zeros $z_{j,n}$ and θ_n of the polynomials $G_n(X)$ have a modulus in the interval*

$$\Big[\, 1 - \frac{2\,\mathrm{Log}\, n}{n},\ 1 + \frac{2\,\mathrm{Log}\, 2}{n}\, \Big], \tag{3.2}$$

(ii) the trinomial $G_n(X)$ admits a unique real root θ_n in the interval $(0,1)$. The sequence $(\theta_n)_{n\geq 2}$ is strictly increasing, $\lim_{n\to+\infty} \theta_n = 1$, with $\theta_2 = \frac{2}{1+\sqrt{5}} = 0.618\ldots$,

(iii) the root θ_n is the unique root of smallest modulus among all the roots of $G_n(X)$; if $n \geq 6$, the roots of modulus < 1 of $G_n(z)$ in the closed upper half-plane have the following properties:

(iii-1) $\theta_n \ < \ |z_{1,n}|$,

(iii-2) for any pair of successive indices $j, j+1$ in $\{1, 2, \ldots, \lfloor n/6 \rfloor\}$,

$$|z_{j,n}| < |z_{j+1,n}|.$$

Proof. (i)(ii) Selmer [Sr], pp 291–292; (iii-1) Flatto, Lagarias and Poonen [FLP], (iii-2) Verger-Gaugry [VG6]. □

The Pisot number (golden mean) $\theta_2^{-1} = \frac{1+\sqrt{5}}{2} = 1.618\ldots$ is the largest Perron number in the family $(\theta_n^{-1})_{n\geq 2}$. The interval $(1, \frac{1+\sqrt{5}}{2}]$ is partitioned by the strictly decreasing sequence of Perron numbers (θ_n^{-1}) as

$$(1, \frac{1+\sqrt{5}}{2}] \ = \ \left(\bigcup_{n=2}^{\infty} [\, \theta_{n+1}^{-1}, \theta_n^{-1}) \right) \bigcup \, \{\theta_2^{-1}\}. \tag{3.3}$$

By the direct method of asymptotic expansions of the roots, as in [VG6], or by Smyth's Theorem [Sy] (Dubickas [Ds]), since the trinomials $G_n(X)$ are not recipro-

cal, the Mahler measure of G_n satisfies

$$\mathrm{M}(\theta_n) \;=\; \mathrm{M}(G_n) \;\geq\; \Theta = 1.3247\ldots, \qquad n \geq 2, \tag{3.4}$$

where $\Theta = \theta_5^{-1}$ is the smallest Pisot number, dominant root of the Pisot polynomial $X^3 - X - 1 = -G_5^*(X)/(X^2 - X + 1)$.

Proposition 3.3. *Let $n \geq 2$. Then (i) the number p_n of roots of $G_n(X)$ which lie inside the open sector $\mathscr{S} = \{z \mid |\arg(z)| < \pi/3\}$ is equal to*

$$1 + 2\lfloor \frac{n}{6} \rfloor, \tag{3.5}$$

(ii) the correlation between the geometry of the roots of $G_n(X)$ which lie inside the unit disc and the upper half-plane and their indexation is given by:

$$j \in \{1, 2, \ldots, \lfloor \frac{n}{6} \rfloor\} \iff \Re(z_{j,n}) > \frac{1}{2} \iff |z_{j,n}| < 1, \tag{3.6}$$

and the Mahler measure $M(G_n)$ of the trinomial $G_n(X)$ is

$$\mathrm{M}(G_n) \;=\; \mathrm{M}(G_n^*) \;=\; \theta_n^{-1} \prod_{j=1}^{\lfloor n/6 \rfloor} |z_{j,n}|^{-2}. \tag{3.7}$$

Proof. Verger-Gaugry [VG6], Proposition 3.7. □

3.2 Asymptotic expansions: roots of G_n and relations

The (Poincaré) asymptotic expansions of the roots of G_n (and G_n^*) are generically written: $\mathrm{Re}(z_{j,n}) = \mathrm{D}(\mathrm{Re}(z_{j,n})) + \mathrm{tl}(\mathrm{Re}(z_{j,n}))$, $\mathrm{Im}(z_{j,n}) = \mathrm{D}(\mathrm{Im}(z_{j,n})) + \mathrm{tl}(\mathrm{Im}(z_{j,n}))$, $\theta_n = \mathrm{D}(\theta_n) + \mathrm{tl}(\theta_n)$, where ”D” and ”tl” stands for *“development”* (or *“limited expansion”*, or *“lowest order terms”*) and ”tl” for *“tail”* (or “remainder”, or *“terminant”* in [Di]). They are given at a sufficiently high order allowing to deduce the asymptotic expansions of the Mahler measures $\mathrm{M}(G_n)$. The terminology *order* comes from the general theory (Borel [Bl], Copson [Cn], Dingle [Di], Erdélyi[E]); the approximant solutions of a polynomial equation say $G(z) = 0$ which arise naturally correspond to *order* 1. The solutions corresponding to *order* 2 are obtained by inserting the *order* 1 *approximant solutions* into the equation $G(z) = 0$, for getting *order* 2 *approximant solutions*. And so on, as a function of $\deg G$. The order is the number of steps in this iterative process. Poincaré [P] introduced this method of divergent series for the N - body problem in celestial mechanics; this method does not appear in number theory in the book “Divergent series” of Hardy. The equivalent of the variable time t (in celestial mechanics) will be the dynamical degree $\mathrm{dyg}(\lceil\!\overline{\alpha}\!\rceil)$ of the house of the algebraic integer α in number theory (with $|\alpha| > 1$), a new “variable concept” introduced in the present study; for the trinomials G_n it will be n.

The asymptotic expansions of θ_n and those roots $z_{j,n}$ of $G_n(z)$ which lie in the first quadrant are (divergent) sums of functions of only *one* variable, which is n, while those of the other roots $z_{j,n}$ are functions of a couple of *two* variables which is:

- $(n, j/n)$ in the angular sector $\pi/4 > \arg z > 2\pi \mathrm{Log}\, n/n$, and
- $(n, j/\mathrm{Log}\, n)$ in the angular sector $2\pi \mathrm{Log}\, n/n > \arg z > 0$.

The first sector is the main angular sector. The second sector if the bump angular sector. A unique regime of asymptotic expansion exists in the main angular sector, whereas two regimes of asymptotic expansions do exist in the bump sector (Appendix, and [VG6]). These regimes are separated by two sequences (u_n) and (v_n), to which the second variable j/n, resp. $j/\mathrm{Log}\, n$, is compared. Details can be found in the Appendix.

Proposition 3.4. *Let $n \geq 2$. The root θ_n can be expressed as: $\theta_n = \mathrm{D}(\theta_n) + \mathrm{tl}(\theta_n)$ with* $\mathrm{D}(\theta_n) = 1-$

$$\frac{\mathrm{Log}\, n}{n}\left(1 - \left(\frac{n - \mathrm{Log}\, n}{n \mathrm{Log}\, n + n - \mathrm{Log}\, n}\right)\left(\mathrm{Log}\,\mathrm{Log}\, n - n\mathrm{Log}\left(1 - \frac{\mathrm{Log}\, n}{n}\right) - \mathrm{Log} n\right)\right) \tag{3.8}$$

and

$$\mathrm{tl}(\theta_n) \;=\; \frac{1}{n} O\left(\left(\frac{\mathrm{Log}\,\mathrm{Log}\, n}{\mathrm{Log}\, n}\right)^2\right), \tag{3.9}$$

with the constant $1/2$ *involved in* $O(\;)$.

Proof. [VG6] Proposition 3.1. □

Lemma 3.5. *Given the limited expansion $D(\theta_n)$ of θ_n as in* (3.8)*, denote*

$$\lambda_n := 1 - (1 - \mathrm{D}(\theta_n))\frac{n}{\mathrm{Log}\, n}.$$

Then $\lambda_n = \mathrm{D}(\lambda_n) + \mathrm{tl}(\lambda_n)$, with

$$\mathrm{D}(\lambda_n) = \frac{\mathrm{Log}\,\mathrm{Log}\, n}{\mathrm{Log}\, n}\left(\frac{1}{1 + \frac{1}{\mathrm{Log}\, n}}\right), \qquad \mathrm{tl}(\lambda_n) = O\left(\frac{\mathrm{Log}\,\mathrm{Log}\, n}{n}\right). \tag{3.10}$$

with the constant 1 in the Big O.

Proof. [VG6] Lemma 3.2. □

In the sequel, for short, we write λ_n instead of $\mathrm{D}(\lambda_n)$.

Proposition 3.6. *Let $n \geq n_0 = 18$ and $1 \leq j \leq \lfloor \frac{n-1}{4} \rfloor$. The roots $z_{j,n}$ of $G_n(X)$ have the following asymptotic expansions: $z_{j,n} = \mathrm{D}(z_{j,n}) + \mathrm{tl}(z_{j,n})$ in the following angular sectors:*

(i) *Sector* $\frac{\pi}{2} > \arg z > 2\pi \frac{\mathrm{Log}\, n}{n}$ *(main sector):*

$$\mathrm{D}(\Re(z_{j,n})) = \cos\left(2\pi \frac{j}{n}\right) + \frac{\mathrm{Log}\left(2 \sin\left(\pi \frac{j}{n}\right)\right)}{n},$$

$$\mathrm{D}(\Im(z_{j,n})) \;=\; \sin\left(2\pi \frac{j}{n}\right) + \tan\left(\pi \frac{j}{n}\right) \frac{\mathrm{Log}\left(2 \sin\left(\pi \frac{j}{n}\right)\right)}{n},$$

with

$$\mathrm{tl}(\Re(z_{j,n})) \;=\; \mathrm{tl}(\Im(z_{j,n})) \;=\; \frac{1}{n} O\left(\left(\frac{\mathrm{Log}\,\mathrm{Log}\, n}{\mathrm{Log}\, n}\right)^2\right)$$

and the constant 1 in the Big O,

(ii) *"Bump" sector* $2\pi \frac{\mathrm{Log}\, n}{n} > \arg z > 0$ *:*

• *Subsector* $2\pi \frac{\sqrt{(\mathrm{Log}\, n)(\mathrm{Log}\,\mathrm{Log}\, n)}}{n} > \arg z > 0$*:*

$$\mathrm{D}(\Re(z_{j,n})) = \theta_n + \frac{2\pi^2}{n} \left(\frac{j}{\mathrm{Log}\, n}\right)^2 \left(1 + 2\lambda_n\right),$$

$$\mathrm{D}(\Im(z_{j,n})) = \frac{2\pi \mathrm{Log}\, n}{n} \left(\frac{j}{\mathrm{Log}\, n}\right) \left[1 - \frac{1}{\mathrm{Log}\, n}(1 + \lambda_n)\right],$$

with

$$\mathrm{tl}(\Re(z_{j,n})) = \frac{1}{n \mathrm{Log}\, n} \left(\frac{j}{\mathrm{Log}\, n}\right)^2 O\left(\left(\frac{\mathrm{Log}\,\mathrm{Log}\, n}{\mathrm{Log}\, n}\right)^2\right),$$

$$\mathrm{tl}(\Im(z_{j,n})) = \frac{1}{n \mathrm{Log}\, n} \left(\frac{j}{\mathrm{Log}\, n}\right) O\left(\left(\frac{\mathrm{Log}\,\mathrm{Log}\, n}{\mathrm{Log}\, n}\right)^2\right),$$

• *Subsector* $2\pi \frac{\mathrm{Log}\, n}{n} > \arg z > 2\pi \frac{\sqrt{(\mathrm{Log}\, n)(\mathrm{Log}\,\mathrm{Log}\, n)}}{n}$*:*

$$\mathrm{D}(\Re(z_{j,n})) = \theta_n + \frac{2\pi^2}{n} \left(\frac{j}{\mathrm{Log}\, n}\right)^2 \left(1 + \frac{2\pi^2}{3} \left(\frac{j}{\mathrm{Log}\, n}\right)^2 (1 + \lambda_n)\right)$$

$$\mathrm{D}(\Im(z_{j,n})) \;=\;$$

$$\frac{2\pi \mathrm{Log}\, n}{n} \left(\frac{j}{\mathrm{Log}\, n}\right) \left[1 - \frac{1}{\mathrm{Log}\, n} \left(1 - \frac{4\pi^2}{3} \left(\frac{j}{\mathrm{Log}\, n}\right)^2 \left(1 - \frac{1}{\mathrm{Log}\, n}(1 - \lambda_n)\right)\right)\right],$$

with

$$\mathrm{tl}(\Re(z_{j,n})) = \frac{1}{n} O\left(\left(\frac{j}{\mathrm{Log}\, n}\right)^6\right), \mathrm{tl}(\Im(z_{j,n})) = \frac{1}{n} O\left(\left(\frac{j}{\mathrm{Log}\, n}\right)^5\right).$$

Proof. [VG6] Proposition 3.4. □

Outside the "bump sector" the moduli of the roots $z_{j,n}$ are readily obtained as (Proposition 3.5 in [VG6]):

$$|z_{j,n}| = 1 + \frac{1}{n}\mathrm{Log}\left(2\sin\left(\frac{\pi j}{n}\right)\right) + \frac{1}{n} O\left(\frac{(\mathrm{Log}\mathrm{Log}\, n)^2}{(\mathrm{Log}\, n)^2}\right), \tag{3.11}$$

with the constant 1 in the Big O (independent of j). The following expansions of the $|z_{j,n}|$s at the order 3 will be needed in the method of Rouché.

Proposition 3.7.

$$\arg(z_{j,n}) = 2\pi\left(\frac{j}{n} + A_{j,n}\right) \quad \textit{with} \quad A_{j,n} = -\frac{1}{2\pi n}\left[\frac{1-\cos(\frac{2\pi j}{n})}{\sin(\frac{2\pi j}{n})}\mathrm{Log}\left(2\sin\left(\frac{\pi j}{n}\right)\right)\right]$$

$$\textit{and} \qquad \mathrm{tl}(\arg(z_{j,n})) = \frac{1}{n} O\left(\left(\frac{\mathrm{Log}\mathrm{Log}\, n}{\mathrm{Log}\, n}\right)^2\right).$$

Proof. §6 in [VG6]. □

Proposition 3.8. *For all j such that $\pi/3 \ge \arg z_{j,n} > 2\pi\frac{\lceil v_n \rceil}{n}$, the asymptotic expansions of the moduli of the roots $z_{j,n}$ are*

$$|z_{j,n}| = \mathrm{D}(|z_{j,n}|) + \mathrm{tl}(|z_{j,n}|)$$

with

$$\mathrm{D}(|z_{j,n}|) = 1 + \frac{1}{n}\mathrm{Log}\left(2\sin\left(\frac{\pi j}{n}\right)\right) + \frac{1}{2n}\left(\frac{\mathrm{Log}\mathrm{Log}\, n}{\mathrm{Log}\, n}\right)^2 \tag{3.12}$$

and

$$\mathrm{tl}(|z_{j,n}|) = \frac{1}{n} O\left(\frac{(\mathrm{Log}\mathrm{Log}\, n)^2}{(\mathrm{Log}\, n)^3}\right) \tag{3.13}$$

where the constant involved in $O(\ )$ is 1 (does not depend upon j).

Proof. [VG6] Section 5.1. □

The following asymptotic expansions in Proposition 3.9, Proposition 3.10 and Proposition 3.11 will be used in the method of Rouché in §5.

Proposition 3.9. *For $n \geq 18$, the modulus of the first root $z_{1,n}$ of $G_n(z) = -1+z+z^n$ is*

$$|z_{1,n}| = 1 - \frac{\operatorname{Log} n - \operatorname{Log}\operatorname{Log} n}{n} + \frac{1}{n} O\left(\frac{\operatorname{Log}\operatorname{Log} n}{\operatorname{Log} n}\right) \tag{3.14}$$

and

$$|-1+z_{1,n}| = \frac{\operatorname{Log} n - \operatorname{Log}\operatorname{Log} n}{n} + \frac{1}{n} O\left(\frac{\operatorname{Log}\operatorname{Log} n}{\operatorname{Log} n}\right) \tag{3.15}$$

with the constant 1 in the two Big Os.

Proof. The root $z_{1,n}$ belongs to the subsector $2\pi \frac{\sqrt{(\operatorname{Log} n)(\operatorname{Log}\operatorname{Log} n)}}{n} > \arg z > 0$: first, from Lemma 3.5, the asymptotic expansion of λ_n is

$$\lambda_n = \frac{\operatorname{Log}\operatorname{Log} n}{\operatorname{Log} n} + O\Big(\frac{\operatorname{Log}\operatorname{Log} n}{(\operatorname{Log} n)^2}\Big)$$

with the constant 1 in the Big O. Since $\mathrm{D}(|z_{1,n}|) = \mathrm{D}(\Re(z_{1,n}))(1 + \big(\frac{\mathrm{D}(\Im(z_{1,n}))}{\mathrm{D}(\Re(z_{1,n}))}\big)^2)^{1/2}$, that

$$\mathrm{D}(\Re(z_{1,n})) = \theta_n + \frac{2\pi^2}{n}\big(\frac{1}{\operatorname{Log} n}\big)^2\big(1+2\lambda_n\big), \mathrm{D}(\Im(z_{1,n})) = \frac{2\pi}{n}\big[1 - \frac{1}{\operatorname{Log} n}(1+\lambda_n)\big]$$

(Proposition 3.6) and

$$\theta_n = 1 - \frac{\operatorname{Log} n}{n}(1-\lambda_n) + \frac{1}{n} O\left(\left(\frac{\operatorname{Log}\operatorname{Log} n}{\operatorname{Log} n}\right)^2\right)$$

(Proposition 3.4) we deduce (3.14) and the expansion (3.15) from the expansion of λ_n. □

Proposition 3.10. *For $n \geq 18$, the modulus of $-1+z_{j,n}$, where $z_{j,n}$ is the j-th root of $G_n(z) = -1+z+z^n$, $\lceil v_n \rceil \leq j \leq \lfloor n/6 \rfloor$, is*

$$|-1+z_{j,n}| \ = \ 2\sin\big(\frac{\pi j}{n}\big) + \frac{1}{n} O\Big(\Big(\frac{\operatorname{Log}\operatorname{Log} n}{\operatorname{Log} n}\Big)^2\Big) \tag{3.16}$$

with the constant 1 in the Big O.

Proof. From (3.11), Proposition 3.6 and Proposition 3.8, the identity

$$|-1+z_{j,n}|^2 = (-1+\Re(z_{j,n}))^2 + (\Im(z_{j,n}))^2 = 1 + |z_{j,n}|^2 - 2\Re(z_{j,n})$$

implies: $|-1+z_{j,n}|^2 =$

$$2 - 2\cos\big(2\pi\frac{j}{n}\big) + \frac{1}{n} O\Big(\Big(\frac{\operatorname{Log}\operatorname{Log} n}{\operatorname{Log} n}\Big)^2\Big) = 4\sin^2\big(\frac{\pi j}{n}\big) + \frac{1}{n} O\Big(\Big(\frac{\operatorname{Log}\operatorname{Log} n}{\operatorname{Log} n}\Big)^2\Big)$$

with the constant 4 in the Big O. We deduce (3.16). □

Proposition 3.11. *For $n \geq 18$, the modulus of $(-1+z_{j,n})/z_{j,n}$, where $z_{j,n}$ is the j-th root of $G_n(z) = -1+z+z^n$, $\lceil v_n \rceil \leq j \leq \lfloor n/6 \rfloor$, is*

$$\frac{|-1+z_{j,n}|}{|z_{j,n}|} = 2\sin(\frac{\pi j}{n})\Big(1 - \frac{1}{n}\mathrm{Log}\,(2\sin(\frac{\pi j}{n}))\Big) + \frac{1}{n}O\left(\left(\frac{\mathrm{Log}\,\mathrm{Log}\,n}{\mathrm{Log}\,n}\right)^2\right) \quad (3.17)$$

with the constant 2 in the Big O.

Proof. The expansion (3.17) readily comes (3.16) and $|z_{j,n}|$ given by Proposition 3.8. □

3.3 Minoration of the Mahler measure

In Verger-Gaugry [VG6] the theory of "à la Poincaré" asymptotic expansions is shown to give "controlled" approximants of the set of the values of the Mahler measures $\mathrm{M}(G_n)$ and an exact value of its limit point. Compared to several methods (Amoroso [A2] [A3], Boyd and Mossinghoff [BM], Dixon and Dubickas [DDs], Langevin [Lg], Smyth [Sy5]), the present approach is new in the sense that the use of auxiliary functions by Dobrowolski [Do2] is replaced by the Rényi-Parry dynamics of the Perron numbers $(\theta_n^{-1})_{n\geq 2}$. Let us briefly mention the results. The product

$$\Pi_{G_n} := D(\mathrm{M}(G_n)) = D(\theta_n)^{-1} \times \prod_{\substack{z_{j,n} \text{ in } |z|<1 \\ \text{outside bump}}} \mathrm{D}(|z_{j,n}|)^{-2} \quad (3.18)$$

is considered, instead of

$$\mathrm{M}(G_n) = \theta_n^{-1} \prod_{j=1}^{\lfloor n/6 \rfloor} |z_{j,n}|^{-2} = \prod_{\mathscr{L}_{\theta_n^{-1}}} |z|^{-1} \quad (3.19)$$

as approximant value of $\mathrm{M}(G_n)$. In (3.18) the zeroes $z_{j,n}$ present in the bump sector are discarded since they do not contribute to the limited asymptotic expansions, as shown in [VG6] Section 4.2. In [VG6] Section 4, the two limits $\lim_{n\to+\infty} \Pi_{G_n}$ and $\lim_{n\to+\infty} \mathrm{M}(G_n)$ are shown to exist, to be equal (and greater than Θ). Theorem 3.12 is obtained either by Boyd-Smyth's method of bivariate Mahler measures ([VG6] Section 4.1) or by the method of asymptotic expansions (Verger-Gaugry [VG6] Section 4.2). The *first derived set* of $\{\mathrm{M}(G_n) \mid n \geq 2\}$ is reduced to one element, $1.38135\ldots$ as follows.

Theorem 3.12. *Let χ_3 be the uniquely specified odd character of conductor* 3 *($\chi_3(m)$ $= 0, 1$ or -1 according to whether $m \equiv 0$, 1 or 2 (mod 3), equivalently $\chi_3(m) = \left(\frac{m}{3}\right)$ the Jacobi symbol), and denote $L(s,\chi_3) = \sum_{m\geq 1} \frac{\chi_3(m)}{m^s}$ the Dirichlet L-series for the character χ_3. Then, with Λ given by* (1.16)*, $\lim_{n\to+\infty} \mathrm{M}(G_n) = \mathrm{M}(-1+z+y) = \Lambda = 1.38135\ldots$*

Proof. [VG6] Theorem 1.1; Smyth [Sy4]. □

Introduced in the product (3.19), the terminants of the asymptotic expansions of the moduli of the roots $z_{j,n}$ and of θ_n provide the higher-order terms of the asymptotic expansion of $\mathrm{M}(G_n)$ as follows.

Theorem 3.13. *Let n_0 be an integer such that $\frac{\pi}{3} > 2\pi \frac{\mathrm{Log}\, n_0}{n_0}$, and let $n \geq n_0$. Then,*

$$\mathrm{M}(G_n) \;=\; \Lambda \left(1 + r(n)\,\frac{1}{\mathrm{Log}\, n} + O\left(\frac{\mathrm{Log}\,\mathrm{Log}\, n}{\mathrm{Log}\, n}\right)^2)\right) \tag{3.20}$$

with the constant $1/6$ involved in the Big O, and with $r(n)$ real, $|r(n)| \leq 1/6$.

Proof. [VG6] Theorem 1.2. □

In Theorem 3.13 we take $n_0 = 18$. For the small values of n, we have:

$$\mathrm{M}(G_2) = \theta_2^{-1} = \frac{1+\sqrt{5}}{2} = 1.618\ldots$$

and the following lower bound.

Proposition 3.14. $\mathrm{M}(G_n) \geq \mathrm{M}(G_5) = \theta_5^{-1} = \Theta = 1.3247\ldots$ *for all $n \geq 3$, with equality if and only if $n = 5$.*

Proof. [VG6] Corollary 1.4. □

The minoration of the residual distance between the two algebraic integers 1 and θ_n^{-1} is deduced from the Zhang-Zagier height and Doche's improvement as follows.

Proposition 3.15. *Let $u = 0$ except if $n \equiv 5 \bmod 6$ in which case $u = -2$. Then,*

$$\mathrm{M}(\theta_n^{-1} - 1) \;\geq\; \frac{\eta^{n+u}}{\Lambda}\big(1 - \frac{1}{6\,\mathrm{Log}\, n}\big), \qquad n \geq 2, \tag{3.21}$$

with $\eta = 1.2817770214$.

Proof. Except for a finite subset of algebraic numbers, the minoration $\mathrm{M}(\alpha)\mathrm{M}(1-\alpha) \geq (\theta_2^{-1/2})^{\deg(\alpha)}$ was established by Zagier [Za] and improved by Doche [Dhe], with the lower bound $\eta > \sqrt{\theta_2^{-1}}$. The minorant (3.21) follows from (2.28) and (3.20). □

The present method of asymptotic expansions gives a new insight into the problem of the minoration of the Mahler measure $\mathrm{M}(G_n)$. Comparing with Dobrowolski's minoration (2.10) [Do2], Theorem 3.13 implies the following minoration of $\mathrm{M}(\theta_n^{-1})$ which is better than (2.10).

Theorem 3.16.

$$\mathrm{M}(\theta_n^{-1}) \; > \; \Lambda - \frac{\Lambda}{6}\left(\frac{1}{\mathrm{Log}\, n}\right), \quad n \geq n_1 = 2. \tag{3.22}$$

Proof. [VG6] Corollary 1.6. □

The extremality of the Perron numbers θ_n^{-1} occurs only for $n = 2, 3$. In general, if extremality holds, by Conjecture 6 (iii), it would be associated with a lenticular distribution of roots (of modulus > 1) which admits a proportion asymptotically equal to $\frac{2}{3}n$. For the trinomials G_n^*, $n \geq 4$, this proportion is only $\frac{1}{3}n$, for n large.

4 The β-shift, Artin-Mazur dynamical zeta function, generalized Fredholm determinant, Perron-Frobenius operator, Parry Upper function, with $\beta > 1$ a real algebraic number

In 1957 Rényi [Re] introduced new representations of a real number x, using a positive function $y = f(x)$ and infinite iterations of it, in the form of an "f-expansion", as

$$x = \varepsilon_0 + f(\varepsilon_1 + f(\varepsilon_2 + \ldots))$$

with "digits" ε_i in some alphabet and remainders $f(\varepsilon_n + f(\varepsilon_{n+1} + \ldots))$. This approach considerably enlarged the usual decimal numeration system, and numeration systems in integer basis, by allowing arbitrary real bases of numeration (Fraenkel [Fl], Lothaire [Lo], Chap. 7): let $\beta > 1$ not an integer and consider $f(x) = x/\beta$ if $0 \leq x \leq \beta$, and $f(x) = 1$ if $\beta < x$. Then the f-expansion of x is the representation of x in base β as

$$x = \varepsilon_0 + \frac{\varepsilon_1}{\beta} + \frac{\varepsilon_2}{\beta^2} + \ldots + \frac{\varepsilon_n}{\beta^n} + \ldots .$$

In terms of dynamical systems, in 1960, Parry [Pa] [Pa2] [Pa3] has reconsidered and studied the ergodic properties of such representations of real numbers in base β, in particular the conditions of faithfullness of the map: $x \longleftrightarrow (\varepsilon_i)_i$ and the complete set of admissible sequences $(\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots)$ for all real numbers (recalled in § 4.1). This complete set is called the language in base β.

In the present note, though the coding of real numbers in arbitrary basis $\beta > 1$ is important in itself, with many generalizations [Ble2] [B-T], [LdM] [BeRo] [PF], [AaPe], [Ro], [Sdt2] [SrTr], it is not the direction we will follow; nor the direction of toral automorphisms and dynamics of Mahler measure [Ld] [Er]. On the contrary we will be mostly interested in the analytical functions associated with the β-shift, i.e. with the language in base β (§ 4.2), $1 < \beta < 2$ first being fixed, and then vary

continuously the basis of numeration β in $\overline{\mathbb{Q}} \cap (1, +\infty)$ taking the limit to 1^+, to use the limit properties of these functions for solving the problem of Lehmer.

The basic analytical function on which relies the solving of the Lehmer's problem is the Parry Upper function $f_\beta(z)$. We will concentrate on its properties, presenting it in a broader context together with the dynamical zeta function $\zeta_\beta(z)$ and the transfer operator $\mathscr{L}_{t\beta}$, their respective literatures being complementary.

4.1 The β-shift, β-expansions, lacunarity and symbolic dynamics

Let $\beta > 1$ be a real number and let $\mathscr{A}_\beta := \{0, 1, 2, \dots, \lceil \beta - 1 \rceil\}$. If β is not an integer, then $\lceil \beta - 1 \rceil = \lfloor \beta \rfloor$. Let x be a real number in the interval $[0, 1]$. A representation in base β (or a β-representation; or a β-ary representation if β is an integer) of x is an infinite word $(x_i)_{i \geq 1}$ of $\mathscr{A}_\beta^{\mathbb{N}}$ such that

$$x = \sum_{i \geq 1} x_i \beta^{-i}.$$

The main difference with the case where β is an integer is that x may have several representations. A particular β-representation, called the β-expansion, or the greedy β-expansion, and denoted by $d_\beta(x)$, of x can be computed either by the greedy algorithm, or equivalently by the β-transformation

$$T_\beta : x \mapsto \beta x \pmod 1 = \{\beta x\}.$$

The dynamical system $([0, 1], T_\beta)$ is called the Rényi-Parry numeration system in base β, the iterates of T_β providing the successive digits x_i of $d_\beta(x)$ [LY]. Denoting $T_\beta^0 := \mathrm{Id}, T_\beta^1 := T_\beta, T_\beta^i := T_\beta(T_\beta^{i-1})$ for all $i \geq 1$, we have:

$$d_\beta(x) = (x_i)_{i \geq 1} \qquad \text{if and only if} \qquad x_i = \lfloor \beta T_\beta^{i-1}(x) \rfloor$$

and we write the β-expansion of x as

$$x = \cdot x_1 x_2 x_3 \dots \qquad \text{instead of} \qquad x = \frac{x_1}{\beta} + \frac{x_2}{\beta^2} + \frac{x_3}{\beta^3} + \dots. \tag{4.1}$$

The digits are $x_1 = \lfloor \beta x \rfloor, x_2 = \lfloor \beta\{\beta x\} \rfloor, x_3 = \lfloor \beta\{\beta\{\beta x\}\} \rfloor, \dots$, depend upon β.

The Rényi-Parry numeration dynamical system in base β allows the coding, as a (positional) β-expansion, of any real number x. Indeed, if $x > 0$, there exists $k \in \mathbb{Z}$ such that $\beta^k \leq x < \beta^{k+1}$. Hence $1/\beta \leq x/\beta^{k+1} < 1$; thus it is enough to deal with representations and β-expansions of numbers in the interval $[1/\beta, 1]$. In the case where $k \geq 1$, the β-expansion of x is

$$x = x_1 x_2 \dots x_k \cdot x_{k+1} x_{k+2} \dots,$$

with $x_1 = \lfloor \beta(x/\beta^{k+1}) \rfloor$, $x_2 = \lfloor \beta\{\beta(x/\beta^{k+1})\} \rfloor$, $x_3 = \lfloor \beta\{\beta\{\beta(x/\beta^{k+1})\}\} \rfloor$, etc. If $x < 0$, by definition: $d_\beta(x) = -d_\beta(-x)$. The part $x_1 x_2 \dots x_k$ is called the β-integer

part of the β-expansion of x, and the terminant $\cdot x_{k+1}x_{k+2}\ldots$ is called the β-fractional part of $d_\beta(x)$.

A β-integer is a real number x such that the β-integer part of $d_\beta(x)$ is equal to $d_\beta(x)$ itself (all the digits x_{k+j} being equal to 0 for $j \geq 1$): in this case, if $x > 0$ for instance, x is the polynomial

$$x = \sum_{i=1}^{k} x_i \beta^{k-i}, \qquad 0 \leq x_i \leq \lceil \beta - 1 \rceil$$

and the set of β-integers is denoted by $\mathbb{Z}_\beta$. For all $\beta > 1$ $\mathbb{Z}_\beta \subset \mathbb{R}$ is discrete and $\mathbb{Z}_\beta = \mathbb{Z}$ if β is an integer $\neq 0, 1$.

The set $\mathscr{A}_\beta^{\mathbb{N}}$ is endowed with the lexicographical order (not usual in number theory), and the product topology. The one-sided shift $\sigma : (x_i)_{i\geq 1} \mapsto (x_{i+1})_{i\geq 1}$ leaves invariant the subset D_β of the β-expansions of real numbers in $[0,1)$. The closure of D_β in $\mathscr{A}_\beta^{\mathbb{N}}$ is called the β-shift, and is denoted by S_β. The β-shift is a subshift of $\mathscr{A}_\beta^{\mathbb{N}}$, for which

$$d_\beta \circ T_\beta = \sigma \circ d_\beta$$

holds on the interval $[0,1]$ (Lothaire [Lo], Lemma 7.2.7). In other terms, S_β is such that

$$x \in [0,1] \qquad \longleftrightarrow \qquad (x_i)_{i\geq 1} \in S_\beta \tag{4.2}$$

is bijective. This one-to-one correspondence between the totally ordered interval $[0,1]$ and the totally lexicographically ordered β-shift S_β is fundamental. Parry ([Pa] Theorem 3) has shown that only one sequence of digits entirely controls the β-shift S_β, and that the ordering is preserved when dealing with the greedy β-expansions. Let us precise how the usual inequality "$<$" on the real line is transformed into the inequality "$<_{lex}$", meaning "lexicographically smaller with all its shifts".
The greatest element of S_β: it comes from $x = 1$ and is given either by the Rényi β-expansion of 1, or by a slight modification of it in case of finiteness. Let us precise it. The greedy β-expansion of 1 is by definition denoted by

$$d_\beta(1) = 0.t_1t_2t_3\ldots \qquad \text{and uniquely corresponds to} \qquad 1 = \sum_{i=1}^{+\infty} t_i \beta^{-i}, \tag{4.3}$$

where

$$t_1 = \lfloor \beta \rfloor, t_2 = \lfloor \beta\{\beta\} \rfloor = \lfloor \beta T_\beta(1) \rfloor, t_3 = \lfloor \beta\{\beta\{\beta\}\} \rfloor = \lfloor \beta T_\beta^2(1) \rfloor, \ldots \tag{4.4}$$

The sequence $(t_i)_{i\geq 1}$ is given by the orbit of one $(T_\beta^j(1))_{j\geq 0}$ by

$$T_\beta^0(1) = 1,\ T_\beta^j(1) = \beta^j - t_1\beta^{j-1} - t_2\beta^{j-2} - \ldots - t_j \in \mathbb{Z}[\beta] \cap [0,1] \tag{4.5}$$

for all $j \geq 1$. The digits t_i belong to $\mathscr{A}_\beta$. We say that $d_\beta(1)$ is finite if it ends in infinitely many zeros.

Definition 4.1. If $d_\beta(1)$ is finite or ultimately periodic (i.e. eventually periodic), then the real number $\beta > 1$ is said to be a *Parry number*. In particular, a Parry number β is said to be *simple* if $d_\beta(1)$ is finite.

The greedy β-expansion of $1/\beta$ is

$$d_\beta(\frac{1}{\beta}) = 0.0t_1t_2t_3\ldots \qquad \text{and uniquely corresponds to} \qquad \frac{1}{\beta} = \sum_{i=1}^{+\infty} t_i \beta^{-i-1}. \quad (4.6)$$

From $(t_i)_{i\geq 1} \in \mathscr{A}_\beta^{\mathbb{N}}$ is built $(c_i)_{i\geq 1} \in \mathscr{A}_\beta^{\mathbb{N}}$, defined by

$$c_1c_2c_3\ldots := \begin{cases} t_1t_2t_3\ldots & \text{if } d_\beta(1) = 0.t_1t_2\ldots \text{ is infinite,} \\ (t_1t_2\ldots t_{q-1}(t_q-1))^\omega & \text{if } d_\beta(1) \text{ is finite, } = 0.t_1t_2\ldots t_q, \end{cases}$$

where $()^\omega$ means that the word within $()$ is indefinitely repeated. The sequence $(c_i)_{i\geq 1}$ is the unique element of $\mathscr{A}_\beta^{\mathbb{N}}$ which allows to obtain all the admissible β-expansions of all the elements of $[0,1)$.

Definition 4.2. A sequence $(y_i)_{i\geq 0}$ of elements of $\mathscr{A}_\beta$ (finite or not) is said admissible if and only if

$$\sigma^j(y_0,y_1,y_2,\ldots) = (y_j,y_{j+1},y_{j+2},\ldots) <_{lex} (c_1,c_2,c_3,\ldots) \quad \text{for all } j \geq 0, \quad (4.7)$$

where $<_{lex}$ means *lexicographically smaller*.

Any admissible representation $(x_i)_{i\geq 1} \in \mathscr{A}_\beta^{\mathbb{N}}$ corresponds, by (4.1), to a real number $x \in [0,1)$ and conversely the greedy β-expansion of x is $(x_i)_{i\geq 1}$ itself. For an infinite admissible sequence $(y_i)_{i\geq 0}$ of elements of $\mathscr{A}_\beta$ the (strict) lexicographical inequalities (4.7) constitute an infinite number of inequalities, unusual in number theory, which are called "Conditions of Parry" [Bd] [Fy] [Fy2] [Lo] [Pa]. More recent criteria of "valid" sequences are considered in Faller and Pfister [FPr].

In number theory, inequalities are often associated to collections of half-spaces in euclidean or adelic Geometry of Numbers (Minkowski's Theorem, etc). The conditions of Parry are of totally different nature since they refer to a reasonable control, order-preserving, of the gappiness (lacunarity) of the coefficient vectors of the power series which are the generalized Fredholm determinants of the transfer operators of the β-transformations.

In the correspondence $[0,1] \longleftrightarrow S_\beta$, the element $x = 1$ admits $d_\beta(1)$ as counterpart. The uniqueness of the β-expansion $d_\beta(1)$ and its self-admissibility property (4.9) characterize the base of numeration β as follows.

Proposition 4.3. *Let $(a_0,a_1,a_2,\ldots)$ be a sequence of non-negative integers where $a_0 \geq 1$ and $a_n \leq a_0$ for all $n \geq 0$. The unique solution $\beta > 1$ of*

$$1 = \frac{a_0}{x} + \frac{a_1}{x^2} + \frac{a_2}{x^3} + \ldots \quad (4.8)$$

is such that $d_\beta(1) = 0.a_0a_1a_2\ldots$ *if and only if*

$$\sigma^n(a_0,a_1,a_2,\ldots) = (a_n,a_{n+1},a_{n+2},\ldots) <_{lex} (a_0,a_1,a_2,\ldots) \qquad \textit{for all } n \geq 1. \quad (4.9)$$

Proof. Corollary 1 of Theorem 3 in Parry [Pa] (Corollary 7.2.10 in Frougny [Fy2]). □

A sequence $(a_i)_{i\geq 0} \in \mathscr{A}_\beta^{\mathbb{N}}$ satisfying (4.9) is said self-admissible. If $1 < \beta < 2$, then the condition "$a_0 \geq 1$ and $a_n \leq a_0$ for all $n \geq 0$" amounts to "$a_0 = 1$"; in this case the β-integer part of β is equal to $a_0 = 1$ and its β-fractional part is $a_1\beta^{-1} + a_2\beta^{-2} + a_3\beta^{-3} + \ldots$. The base of numeration $\beta = 1$ would correspond to the sequence $(1,0,0,0,\ldots)$ in (4.8) but this sequence has its first digit 1 outside the alphabet $\mathscr{A}_1 = \{0\}$: it cannot be considered as a 1-expansion. Fortunately numeration in base one is not often used. The base of numeration $\beta = 2$ would correspond to the constant sequence $(1,1,1,1,\ldots)$ in (4.8) but this sequence is not self-admissible. When $\beta = 2$, 2 being an integer, 2-ary representations differ and $(2,0,0,0,\ldots)$ is taken instead of $(1,1,1,1,\ldots)$ (Frougny and Sakarovitch [FySh], Lothaire [Lo]).

Infinitely many cases of lacunarity, between $(1,0,0,0,\ldots)$ and $(1,1,1,1,\ldots)$, may occur in the sequence $(a_0,a_1,a_2,\ldots)$ in (4.8). If $\beta \in (1,2)$ is fixed, with $d_\beta(1) = 0.t_1t_2t_3\ldots$ then any x, $1/\beta < x < 1$, admits a β-expansion $d_\beta(x)$ which lies lexicographically (Parry [Pa], Lemma 1) between those of the extremities:

$$d_\beta(\frac{1}{\beta}) = 0.0t_1t_2t_3\ldots \quad <_{lex} \quad d_\beta(x) \quad <_{lex} \quad d_\beta(1) = 0.t_1t_2t_3\ldots. \qquad (4.10)$$

Let $1 < \beta < 2$ be a real number, with $d_\beta(1) = 0.t_1t_2t_3\ldots$. If β is a simple Parry number, then there exists $n \geq 2$, depending upon β, such that $t_n \neq 0$ and $t_j = 0, j \geq n+1$. Parry [Pa] has shown that the set of simple Parry numbers is dense in the half-line $(1,+\infty)$. If β is a Parry number which is not simple, the sequence $(t_i)_{i\geq 1}$ is eventually periodic: there exists an integer $m \geq 1$, the preperiod length, and an integer $p \geq 1$, the period length, such that

$$d_\beta(1) = 0.t_1t_2\ldots t_m(t_{m+1}t_{m+2}\ldots t_{m+p})^\omega,$$

m and p depending upon β, with at least one nonzero digit t_j, with $j \in \{m+1, m+2,\ldots,m+p\}$. The gaps of successive zeroes in $(t_i)_{i\geq 1}$ are those of the preperiod $(t_1,t_2,\ldots,t_m)$ then those of the period $(t_{m+1},t_{m+2},\ldots,t_{m+p})$, then occur periodically up till infinity. The length of such gaps of zeroes is at most $\max\{m-2, p-1\}$. The asymptotic lacunarity is controlled by the periodicity in this case.

If $1 < \beta < 2$ is an algebraic number which is not a Parry number, the sequences of gaps of zeroes in $(t_i)_{i\geq 1}$ remain asymptotically moderate and controlled by the Mahler measure $\mathrm{M}(\beta)$ of β, as follows.

Theorem 4.4 (Verger-Gaugry). *Let* $\beta > 1$ *be an algebraic number such that* $d_\beta(1)$ *is infinite and gappy in the sense that there exist two infinite sequences* $\{m_n\}_{n\geq 1}$ *and*

$\{s_n\}_{n\geq 0}$ *such that*

$$1 = s_0 \leq m_1 < s_1 \leq m_2 < s_2 \leq \ldots \leq m_n < s_n \leq m_{n+1} < s_{n+1} \leq \ldots$$

with $(s_n - m_n) \geq 2$*,* $t_{m_n} \neq 0$*,* $t_{s_n} \neq 0$ *and* $t_i = 0$ *if* $m_n < i < s_n$ *for all* $n \geq 1$*. Then*

$$\limsup_{n\to+\infty} \frac{s_n}{m_n} \leq \frac{\operatorname{Log}(\mathrm{M}(\beta))}{\operatorname{Log}\beta} \tag{4.11}$$

Proof. [VG], Theorem 1.1. □

Theorem 4.4 also became a consequence of Theorem 2 in [AiB]. In Theorem 4.4 the quotient $s_n/m_n, n \geq 1$, is called the n-th Ostrowski quotient of the sequence $(t_i)_{i\geq 1}$. For a given algebraic number $\beta > 1$, whether the upper bound (4.11) is exactly the limsup of the sequence of the Ostrowski quotient of $(t_i)_{i\geq 1}$ is unknown. For Salem numbers, this equality always holds since $\mathrm{M}(\beta) = \beta$, and the upper bound (4.11) is 1.

Varying the base of numeration β *in the interval* $(1,2)$*:* for all $\beta \in (1,2)$, being an algebraic number or a transcendental number, the alphabet $\mathscr{A}_\beta$ of the β-shift is always the same: $\{0,1\}$. All the digits of all β-expansions $d_\beta(1)$ are zeroes or ones. Parry ([Pa]) has proved that the relation of order $1 < \alpha < \beta < 2$ is preserved on the corresponding greedy α- and β- expansions $d_\alpha(1)$ and $d_\beta(1)$ as follows.

Proposition 4.5. *Let* $\alpha > 1$ *and* $\beta > 1$*. If the Rényi* α*-expansion of 1 is*

$$d_\alpha(1) = 0.t'_1 t'_2 t'_3 \ldots, \qquad i.e. \quad 1 = \frac{t'_1}{\alpha} + \frac{t'_2}{\alpha^2} + \frac{t'_3}{\alpha^3} + \ldots$$

and the Rényi β*-expansion of 1 is*

$$d_\beta(1) = 0.t_1 t_2 t_3 \ldots, \qquad i.e. \quad 1 = \frac{t_1}{\beta} + \frac{t_2}{\beta^2} + \frac{t_3}{\beta^3} + \ldots,$$

then $\alpha < \beta$ *if and only if* $(t'_1, t'_2, t'_3, \ldots) <_{lex} (t_1, t_2, t_3, \ldots)$*.*

Proof. Lemma 3 in Parry [Pa]. □

For any integer $n \geq 1$ the sequence of digits $10^{n-1}1$, with $n-1$ times "0" between the two ones, is self-admissible. By Proposition 4.3 it defines an unique solution $\beta \in (1,2)$ of (4.8). Denote by θ_{n+1}^{-1} this solution. From Proposition 4.5 we deduce that the sequence $(\theta_n^{-1})_{n\geq 2}$ is (strictly) decreasing and tends to 1 when n tends to infinity.

From (4.8) the real number θ_2^{-1} is the unique root > 1 of the equation $1 = 1/x + 1/x^2$, that is of $X^2 - X - 1$. Therefore it is the Pisot number (golden mean) $= \frac{1+\sqrt{5}}{2} = 1.618\ldots$. Being interested in bases $\beta > 1$ close to 1 tending to 1^+, we will focus on the interval $(1, \frac{1+\sqrt{5}}{2}]$ in the sequel. This interval is partitioned by the decreasing

sequence $(\theta_n^{-1})_{n\geq 2}$ as

$$\big(1, \frac{1+\sqrt{5}}{2}\big] \;=\; \bigcup_{n=2}^{\infty} \big[\,\theta_{n+1}^{-1}, \theta_n^{-1}\big) \;\bigcup\; \{\theta_2^{-1}\}. \tag{4.12}$$

Theorem 4.4 gives an upper bound of the asymptotic behaviour of the Ostrowski quotients of the β-expansion $(t_i)_{i\geq 1}$ of 1, due to the fact that $\beta > 1$ is an algebraic number. The following theorem shows that the gappiness of $(t_i)_{i\geq 1}$ also admits some uniform lower bound, for all gaps of zeroes. The condition of minimality on the length of the gaps of zeroes in $(t_i)_{i\geq 1}$ is only a function of the interval $\big[\,\theta_{n+1}^{-1}, \theta_n^{-1}\big)$ to which β belongs, when β tends to 1.

Theorem 4.6. *Let $n \geq 2$. A real number $\beta \in (1, \frac{1+\sqrt{5}}{2}]$ belongs to $[\theta_{n+1}^{-1}, \theta_n^{-1})$ if and only if the Rényi β-expansion of unity is of the form*

$$d_\beta(1) = 0.10^{n-1}10^{n_1}10^{n_2}10^{n_3}\ldots, \tag{4.13}$$

with $n_k \geq n-1$ for all $k \geq 1$.

Proof. Since $d_{\theta_{n+1}^{-1}}(1) = 0.10^{n-1}1$ and $d_{\theta_n^{-1}}(1) = 0.10^{n-2}1$, Proposition 4.5 implies that the condition is sufficient. It is also necessary: $d_\beta(1)$ begins as $0.10^{n-1}1$ for all β such that $\theta_{n+1}^{-1} \leq \beta < \theta_n^{-1}$. For such βs we write $d_\beta(1) = 0.10^{n-1}1u$ with digits in the alphabet $\mathscr{A}_\beta = \{0,1\}$ common to all βs, that is

$$u = 1^{h_0}0^{n_1}1^{h_1}0^{n_2}1^{h_2}\ldots$$

and $h_0, n_1, h_1, n_2, h_2, \ldots$ integers ≥ 0. The self-admissibility lexicographic condition (4.9) applied to the sequence $(1, 0^{n-1}, 1^{1+h_0}, 0^{n_1}, 1^{h_1}, 0^{n_2}, 1^{h_3}, \ldots)$, which characterizes uniquely the base of numeration β, readily implies $h_0 = 0$ and $h_k = 1$ and $n_k \geq n-1$ for all $k \geq 1$. □

Remark 4.7. The case $n_1 = +\infty$ in (4.13) corresponds to the simple Parry number $\beta = \theta_{n+1}^{-1}$. The value $+\infty$ is not excluded from the set $(n_k)_{k\geq 1}$ in the following sense: if there exists $j \geq 2$ such that $n-1 \leq n_k < +\infty$, $k < j$, with $n_j = +\infty$, then β is a simple Parry number in $[\theta_{n+1}^{-1}, \theta_n^{-1})$ characterized by

$$d_\beta(1) = 0.10^{n-1}10^{n_1}10^{n_2}1\ldots 10^{n_{j-1}}1,$$

that is is a root of an integer polynomial (§ 4.3). All the simple Parry numbers lying in the interval $[\theta_{n+1}^{-1}, \theta_n^{-1})$ are obtained in this way. On the contrary, the transcendental numbers β in $[\theta_{n+1}^{-1}, \theta_n^{-1})$ have all Rényi β-expansions $d_\beta(1) = 0.t_1t_2t_3\ldots$ of 1 such that the sequence of exponents $(n_k)_{k\geq 1}$ of the successive zeroes, corresponding to the sequence of the lengths of the gaps of zeroes, never takes the value $+\infty$.

Definition 4.8. Let $\beta \in (1, \frac{1+\sqrt{5}}{2}]$ be a real number. The integer $n \geq 3$ such that $\theta_n^{-1} \leq \beta < \theta_{n-1}^{-1}$ is called the dynamical degree of β, and is denoted by $\mathrm{dyg}(\beta)$. By convention we put: $\mathrm{dyg}(\frac{1+\sqrt{5}}{2}) = 2$.

The function $n = \mathrm{dyg}(\beta)$ is locally constant on the interval $(1, \frac{1+\sqrt{5}}{2}]$, is decreasing, takes all values in $\mathbb{N} \setminus \{0,1\}$, and satisfies: $\lim_{\beta>1, \beta\to 1} \mathrm{dyg}(\beta) = +\infty$. The relations between the restriction of the dynamical degree $\mathrm{dyg}(\beta)$ to $\overline{\mathbb{Q}} \cap (1, \frac{1+\sqrt{5}}{2}]$ and the (usual) degree $\deg(\beta)$ will be investigated later (Theorem 1.1; § 5.1, § 6.5). Let us observe that the equality $\deg(\beta) = \mathrm{dyg}(\beta) = 2$ holds if $\beta = \frac{1+\sqrt{5}}{2}$, but the equality case is not the case in general.

Definition 4.9. A power series $\sum_{j=0}^{+\infty} a_j z^j$, with $a_j \in \{0,1\}$ for all $j \geq 0$, z the complex variable, is said to be self-admissible if its coefficient vector $(a_i)_{i\geq 0}$ is self-admissible.

4.2 Parry Upper functions, Dynamical zeta functions and transfer operators

Definition 4.10. Let $\beta \in (1, (1+\sqrt{5})/2]$ be a real number, and $d_\beta(1) = 0.t_1t_2t_3\ldots$ its Rényi β-expansion of 1. The power series $f_\beta(z) := -1 + \sum_{i\geq 1} t_i z^i$ of the complex variable z is called the Parry Upper function at β.

By definition of $d_\beta(1) = 0.t_1t_2t_3\ldots$ the Parry Upper function $f_\beta(z)$ has a zero at $z = 1/\beta$. It is such that $f_\beta(z) + 1$ has coefficients in the alphabet $\mathscr{A}_\beta = \{0,1\}$ and is self-admissible by (4.9); if β is an algebraic number in particular, the lacunarity of $f_\beta(z)$ is moderate by Theorem 4.4, because of the existence of an upper bound of the Ostrowski quotients of $(t_i)_{i\geq 0}$ in terms of the Mahler measure $\mathrm{M}(\beta)$, and by Theorem 4.6, by the dynamical degree $\mathrm{dyg}(\beta)$. Since $d_\beta(1) = 0.t_1t_2t_3\ldots$ entirely determines the language in base β, the function $f_\beta(z)$ is an analytic function whose domain of definition and zeroes are analytical characteristics of the language in base β. Therefore some continuity properties of some subcollections of zeroes are expected, when β runs over a small neighbourhood of 1 (§ 4.4).

The definition of $f_\beta(z)$ seems simple since the vector coefficient of $f_\beta(z) + 1$ is only a sequence of integers deduced from the orbit of 1 under the iterates of the β-transformation T_β, by (4.4) and (4.5) [FL], I, [FLP]; nevertheless it is deeply related to the Artin-Mazur dynamical zeta function $\zeta_\beta(z)$ (given by (1.25)) of the Rényi-Parry dynamical system $([0,1], T_\beta)$, to the Perron-Frobenius operator P_{T_β} associated with T_β and to the generalized "Fredholm determinant" (4.15) of this operator, or more precisely of the transfer operator of T_β. In the kneading theory of Milnor and Thurston [MorT] it is a kneading determinant. Let us recall these links, knowing that the theory of Fredholm (Grothendieck [Gr] [Gr2], Riesz and Nagy [RSN] Chap. IV) is done for compact operators while the Perron-Frobenius operators associated with

the β-transformations T_β are noncompact by nature (Mori [Mo] [Mo2], Takahashi [T] [T2] [T5]).

Let (X,Σ,μ) be a σ-finite measure space and let $T : X \to X$ be a nonsingular transformation, i.e. T is measurable and satisfies: for all $A \in \Sigma$, $\mu(A) = 0 \Longrightarrow \mu(T^{-1}(A)) = 0$. In ergodic theory, by the Radon-Nikodym theorem, the operator $P_T : L^1(X,\Sigma,\mu) \to L^1(X,\Sigma,\mu)$ defined by

$$\int_A P_T f d\mu \;=\; \int_{T^{-1}(A)} f d\mu \tag{4.14}$$

is called the Perron-Frobenius operator associated with T. Let $\beta \in (1,\theta_2^{-1})$, $X = [0,1]$, Σ the Borel σ-algebra and T_β the β-transformation. The T_β-invariant probability measure $\mu = \mu_\beta$ of the β-shift, on Σ, is unique (Rényi [Re]), ergodic (Parry [Pa]), maximal (Hofbauer [Hr]) and absolutely continuous with respect to the Lebesgue measure dt, with Radon-Nikodym derivative (Lasota and Yorke [LY], Parry [Pa], Takahashi [T]) :

$$h_\beta \;=\; C \sum_{n: x < T_\beta^n(1)} \frac{1}{\beta^{n+1}}, \qquad \text{so that} \qquad d\mu_\beta = h_\beta dt,$$

for some constant $C > 0$. These results were independently discovered by A.O. Gelfond [FPr]. We denote by P_{T_β} the Perron-Frobenius operator associated with T_β.

The β-transformation T_β is a piecewise monotone map of the interval $[0,1]$ with weight function $g = 1$. Over the last 45 years, the rigourous theory of dynamical zeta functions, in particular weighted dynamical zeta functions, introduced by Ruelle [Ru] [Ru2] [Ru3] [Ru4] [Ru5] [Ru6] [Ru7] [Ru8] [Ru9] by analogy with the thermodynamic formalism of statistical mechanism [Ru4], settled as a basic analytic theory for the study of many dynamical systems. The relations between the Fredholm determinants, or the generalized Fredholm determinants, and these weighted dynamical zeta functions, were extensively studied (Baladi [Ba] [Ba2] [Ba3], Baladi and Keller [BaK], Hofbauer [Hr], Hofbauer and Keller [HrK], Milnor and Thurston [MorT], Parry and Pollicott [PaPt], Pollicott [Pt] [Pt2], Preston [Pn], Takahashi [T3] [T4]). In the context of noncompact operators, the objective consists in giving a sense to

$$\text{`det`}\,(Id - zL_t) = \exp\Big(-\sum_{n\geq 1} \frac{\text{`}tr\text{`}\,L_t^n}{n}\, z^n\Big), \tag{4.15}$$

where L_t is a dynamically defined weighted transfer operator acting on a suitable Banach space [Ba2] [HrK2].

Let $1 < \beta < (1+\sqrt{5})/2$ be a real number and $0 = a_0 < a_1 = \frac{1}{\beta} < a_2 = 1$ be the finite partition of $[0,1]$. The map T_β is strictly monotone and continuous on $[a_0,a_1)$ and $[a_1,1]$. For each function $f : [0,1] \to \mathbb{C}$, let

$$\operatorname{var}(f) := \sup\Big\{ \sum_{i=1}^{n} |f(e_i) - f(e_{i-1})| \mid n \geq 1, 0 \leq e_1 \leq e_2 \leq \ldots \leq e_n \leq 1 \Big\},$$

$$\|f\|_{BV} := \mathrm{var}(f) + \sup(|f|),$$

and denote by BV the Banach space of functions with bounded variation [K] [K2]:

$$BV := \{f : [0,1] \to \mathbb{C} \mid \|f\|_{BV} < \infty\}.$$

For $g \in BV$, one can define the following transfer operator

$$L_{t\beta,g} : BV \to BV, \quad L_{t\beta,g} f(x) := \sum_{y, T_\beta(y)=x} g(y) f(y).$$

We will only consider the case $g \equiv 1$ in the sequel and put $L_{t\beta} := L_{t\beta,1}$.

Theorem 4.11. *Let $\beta \in (1, \theta_2^{-1})$. Then,*

(i) *the Artin-Mazur dynamical zeta function $\zeta_\beta(z)$ defined by* (1.25) *is nonzero and meromorphic in $\{|z| < 1\}$, and such that $1/\zeta_\beta(z)$ is holomorphic in $\{|z| < 1\}$,*

(ii) *suppose $|z| < 1$. Then z is a pole of $\zeta_\beta(z)$ of multiplicity k if and only if z^{-1} is an eigenvalue of $L_{t\beta}$ of multiplicity k.*

Proof. Theorem 2 in Baladi and Keller [BaK], assuming that the set of intervals $([0,a_1),[a_1,1])$ forming the partition of $[0,1]$ is generating; In [Ru5] [Ru8] Ruelle shows that this assumption is not necessary, showing how to remove this obstruction. □

The β-transformation T_β is one of the simplest transformations among piecewise monotone intervals maps (Baladi and Ruelle [BaR], Milnor and Thurston [MorT], Pollicott [Pt]). Theorem 4.11 was conjectured by Hofbauer and Keller [HrK] for piecewise monotone maps, for the case where the function g is piecewise constant. (cf also Mori [Mo] [Mo2]). The case $g = 1$ in the transfer operators was studied by Milnor and Thurston [MorT], Hofbauer [Hr], Preston [Pn]. The structure of the sets of periodic points of T_β are studied as Markov diagrams by Hofbauer [Hr2].

The eigenvalues of the transfer operators, not only of the transfer operators $L_{t\beta}$ with $1 < \beta < \theta_2^{-1}$, are important quantities, for instance for resonances in dynamical systems, decays of correlations [Ba4] [En]. Theorem 4.11 was proved and stated in Baladi and Keller [BaK] under more general assumptions.

The fact (Theorem 4.11 (ii)) that the poles of $\zeta_\beta(z)$, lying in the open unit disc, are of the same multiplicity of the inverses of the eigenvalues of the transfer operator $L_{t\beta}$ is a extension of Theorem 2, Theorem 3 and Theorem 4 in Grothendieck [Gr2] in the context of the Fredholm theory with compact operators. When $\beta > 1$ is an algebraic integer and tends to 1^+, we will prove (§ 5) that the multiplicity k is equal to 1 for the first pole $1/\beta$ of $\zeta_\beta(z)$ and for a subcollection of Galois conjugates of $1/\beta$ in an angular sector.

The relations between the poles of the dynamical zeta function $\zeta_\beta(z)$, the zeroes of the Parry Upper function $f_\beta(z)$ and the eigenvalues of the transfer operator $L_{t\beta}$ come from Theorem 4.11 and from the following theorem.

Theorem 4.12. *Let $\beta > 1$ be a real number. Then the Parry Upper function $f_\beta(z)$ satisfies*

$$(i) \qquad f_\beta(z) = -\frac{1}{\zeta_\beta(z)} \qquad \text{if } \beta \text{ is not a simple Parry number}, \tag{4.16}$$

and

$$(ii) \quad f_\beta(z) = -\frac{1-z^N}{\zeta_\beta(z)} \qquad \text{if } \beta \text{ is a simple Parry number} \tag{4.17}$$

where N, which depends upon β, is the minimal positive integer such that $T_\beta^N(1) = 0$. It is holomorphic in the open unit disk $\{|z| < 1\}$. It has no zero in $|z| \le 1/\beta$ except $z = 1/\beta$ which is a simple zero. The Taylor series of $f_\beta(z)$ at $z = 1/\beta$ is $f_\beta(z) = c_{\beta,1}\big(z - \frac{1}{\beta}\big) + c_{\beta,2}\big(z - \frac{1}{\beta}\big)^2 + \ldots$ with

$$c_{\beta,m} = \sum_{n=m}^{\infty} \frac{n!}{(n-m)!\; m!} \lfloor \beta T_\beta^{n-1}(1) \rfloor \big(\frac{1}{\beta}\big)^{n-m} > 0, \qquad \text{for all } m \ge 1. \tag{4.18}$$

Proof. Theorem 2.3 and Appendix A in Flatto, Lagarias and Poonen [FLP]; Theorem 1.2 in Flatto and Lagarias [FL], I; Theorem 3.2 in Lagarias [Ls]. From Takahashi [T], Ito and Takahashi [IT], these authors deduce

$$\zeta_\beta(z) = \frac{1-z^N}{(1-\beta z)\Big(\sum_{n=0}^{\infty} T_\beta^n(1)\, z^n\Big)} \tag{4.19}$$

where "z^N" has to be replaced by "0" if β is not a simple Parry number. Since $\beta T_\beta^n(1) = \lfloor \beta T_\beta^n(1) \rfloor + \{\beta T_\beta^n(1)\} = t_{n+1} + T_\beta^{n+1}(1)$ by (4.4), for $n \ge 1$, expanding the power series of the denominator (4.19) readily gives:

$$-1 + t_1 z + t_2 z^2 + \ldots = f_\beta(z) = -(1-\beta z)\Big(\sum_{n=0}^{\infty} T_\beta^n(1)\, z^n\Big). \tag{4.20}$$

The zeroes of smallest modulus are characterized in Lemma 5.2, Lemma 5.3 and Lemma 5.4 in [FLP]. The coefficients $c_{\beta,m}$ readily come from the derivatives of $f_\beta(z)$. □

The set of Parry numbers $\mathbb{P}_P$ is not characterized (§ 4.3). Case (i) means that $\zeta_\beta^{-1}(z)$, holomorphic in $|z| < 1$, is a power series with coefficients in a finite set of complex numbers, for β being a nonParry number. As an appreciable advantage, in the simple Parry number case, case (ii) means that there exists a product of cyclotomic polynomials, namely $-(1-z^N)$, which is such that its product with $\zeta_\beta^{-1}(z)$, does the same: $f_\beta(z)$ is also a power series with coefficients in a finite set. Then, since the roots of cyclotomic polynomials are of modulus 1, the zeroes of $f_\beta(z)$ within $|z| < 1$ are exactly the poles of $\zeta_\beta(z)$ in this domain, whatever the Rényi-Parry dynamics of $\beta > 1$ is.

The Parry Upper function $f_\beta(z)$ admits an Euler product and a logarithmic derivative in terms of a power series with coefficients in $\mathbb{Z}$: from (1.25) and [FL], I,

$$\zeta_\beta(z) = \prod_{x \text{ periodic}} \left(1 - z^{\text{order}(x)}\right)^{-1}, \qquad \frac{\zeta_\beta'(z)}{\zeta_\beta(z)} = \sum_{n=0}^{\infty} \mathscr{P}_{n+1} z^n$$

and from (4.16) (4.17), with N defined inthere, we deduce, with "z^N" equal to 0 if β is not a simple Parry number,

$$f_\beta(z) = -(1 - z^N) \times \prod_{x \text{ periodic}} \left(1 - z^{\text{order }(x)}\right), \tag{4.21}$$

$$\frac{f_\beta'(z)}{f_\beta(z)} = -N z^{N-1}(1 - z^N)^{-1} - \sum_{n=0}^{\infty} \mathscr{P}_{n+1} z^n. \tag{4.22}$$

The growth of the sequence $(\mathscr{P}_n)_{n \geq 0}$ was investigated in Flatto and Lagarias [FL] in terms of the lap-function of the β-shift as a function of the two zeroes of smallest modulus of $f_\beta(z)$.

Definition 4.13. If β is a simple Parry number, with $d_\beta(1) = 0.t_1 t_2 \ldots t_m$, $t_m \neq 0$, the polynomial

$$P_{\beta,P}(X) := X^m - t_1 X^{m-1} - t_2 X^{m-2} - \ldots t_m \tag{4.23}$$

is called the Parry polynomial of β. If β is a Parry number which is not simple, with $d_\beta(1) = 0.t_1 t_2 \ldots t_m (t_{m+1} t_{m+2} \ldots t_{m+p+1})^\omega$ and not purely periodic (m is $\neq 0$), then

$$\begin{aligned} P_{\beta,P}(X) := X^{m+p+1} - t_1 X^{m+p} - t_2 X^{m+p-1} - \ldots - t_{m+p} X - t_{m+p+1} \\ - X^m + t_1 X^{m-1} + t_2 X^{m-2} + \ldots + t_{m-1} X + t_m \end{aligned} \tag{4.24}$$

is the Parry polynomial of β. If β is a nonsimple Parry number such that $d_\beta(1) = 0.(t_1 t_2 \ldots t_{p+1})^\omega$ is purely periodic (i.e. $m = 0$), then

$$P_{\beta,P}(X) := X^{p+1} - t_1 X^p - t_2 X^{p-1} - \ldots - t_p X - (1 + t_{p+1}) \tag{4.25}$$

is the Parry polynomial of β. By definition the degree d_P of $P_{\beta,P}(X)$ is respectively $m, m+p-1, p+1$ in the three cases.

If β is a Parry number, the Parry polynomial $P_{\beta,P}(X)$, belonging to the ideal $P_\beta(X)\mathbb{Z}[X]$, admits β as simple root and is often not irreducible. Its factorization properties were studied in the context of the theory of Pinner and Vaaler [PrVr] in [VG2]; see also [BMs2]. The polynomial $\frac{P_{\beta,P}(X)}{P_\beta(X)}$ was called *complementary factor* by Boyd. For the two cases (4.23) and (4.25) the constant term is $\neq 0$; hence $\deg(P^*_{\beta,P}) = \deg(P_{\beta,P})$. In the case of (4.24) denote by $q_\beta := 0$ if $t_m \neq t_{m+p+1}$ and, if $t_m = t_{m+p+1}$,

$q_\beta := 1 + \max\{r \in \{0, 1, m-1\} \mid t_{m-l} = t_{m+p+1-l}$ for all $0 \le l \le r\}$. Then $p+1 \le \deg(P_{\beta,P}) - q_\beta = \deg(P^*_{\beta,P}) \le \deg(P_{\beta,P})$.

Applying the Carlson-Polya dichotomy (Bell and Chen [BCn], Bell, Miles and Ward [BMW], Carlson [C] [C2], Dienes [Dis], Pólya [Pl], Robinson [Ron], Szegő [Szo]) to the power series $f_\beta(z)$, for which the coefficients belong to the finite set $\mathscr{A}_\beta \cup \{-1\}$, gives the following equivalence.

Theorem 4.14. *The real number $\beta > 1$ is a Parry number if and only if the Parry Upper function $f_\beta(z)$ is a rational function, equivalently if and only if $\zeta_\beta(z)$ is a rational function.*

The set of Parry numbers, resp. of nonParry numbers, in $(1, \infty)$, is not empty. If β is not a Parry number, then $|z| = 1$ is the natural boundary of $f_\beta(z)$. If β is a Parry number, with Rényi β-expansion of 1 given by

$$d_\beta(1) = 0.t_1 t_2 \ldots t_m (t_{m+1} t_{m+2} \ldots t_{m+p+1})^\omega, \qquad t_1 = \lfloor \beta \rfloor,\ t_i \in \mathscr{A}_\beta, i \ge 2,$$

the preperiod length being $m \ge 0$ and the period length $p+1 \ge 1$, $f_\beta(z)$ admits an analytic meromorphic extension over $\mathbb{C}$, of the following form:

$$f_\beta(z) = -P^*_{\beta,P}(z) \qquad \text{if } \beta \text{ is simple,}$$

$$f_\beta(z) = \frac{-P^*_{\beta,P}(z)}{1 - z^{p+1}} \qquad \text{if } \beta \text{ is nonsimple,}$$

where the Parry polynomial is given by (4.23), (4.24) *or* (4.25).

If $\beta \in (1,2)$ is a Parry number, the (naïve) height $\mathrm{H}(P_{\beta,P})$ *of $P_{\beta,P}$ is equal to 1 except when: β is nonsimple and that $t_{p+1} = \lfloor \beta T^p_\beta(1) \rfloor = 1$, in which case the Parry polynomial of β has naïve height* $\mathrm{H}(P_{\beta,P}) = 2$.

Proof. Verger-Gaugry [VG6]. The set of nonParry numbers β in $(1, \infty)$ is not empty as a consequence of Fekete-Szegő's Theorem [FeSo] since the radius of convergence of $f_\beta(z)$ is equal to 1 in any case and that its domain of definition always contains the open unit disk which has a transfinite diameter equal to 1. The set of Parry numbers β in $(1, \infty)$ is also nonempty. Indeed Pisot numbers, of degree ≥ 2, are Parry numbers (Schmidt [Sdt], Bertrand-Mathis [BMs]). Therefore the dichotomy between Parry and nonParry numbers in $(1, \infty)$ has a sense. □

Definition 4.15. Let $\beta > 1$ be a Parry number. If the Parry polynomial $P_{\beta,P}(z)$ of β is not irreducible, the roots of $P_{\beta,P}(z)$ which are not Galois conjugates of β are called the beta-conjugates of β.

Beta-conjugates were studied in [VG4] [VG5] in terms of Puiseux theory and in association with germs of curves.

Remark 4.16. For any Parry number $\beta \in (1,2)$ the (naïve) height $\mathrm{H}(P_{\beta,P})$ of the Parry polynomial $P_{\beta,P}(z)$ is uniformly bounded by 2, by (4.23) (4.24) (4.25). Now let

us consider a convergent sequence of Parry numbers (β_i), $\beta_i < 2$, tending to 1. This remarkable property has for consequence that the phenomenon of limit equidistribution towards the Haar measure on the unit circle occurs. Here, "limit equidistribution" means that the set constituted by the conjugates and the beta-conjugates of the β_is, both denoted by $\beta_i^{(j)}$, tend to the unit circle for the Hausdorff topology (Theorem 3.25 in [VG3]). The basic argument is the tightness property (Billingsley [Biy]) of the convergent sequence

$$\frac{1}{[\mathbb{K}:\mathbb{Q}]}\sum \delta_{\beta_i^{(j)}} \qquad i \to \infty,$$

for the weak topology, which is then satisfied, following Pritsker [Pr] [Pr3] (cf also § 8). Removing the subcollections of beta-conjugates, and leaving only the conjugates of the Parry numbers β_i, may suppress the phenomenon of limit equidistribution.

4.2.1 Algebraicity of the base and zeroes of Parry Upper functions When β is a Parry number, the Parry Upper function $f_\beta(z)$ has a finite number of zeroes by Theorem 4.14. If β is not a Parry number the presence of zeroes of $f_\beta(z)$ in $1/\beta < |z| < 1$ is more difficult to describe. The most important region to be investigated for the presence, the number (eventually infinite) and the geometry of zeroes of $f_\beta(z)$ is the annular region $\{z \mid \exp(-\mathscr{H}) = 1/\beta < |z| < 1\}$, where $\mathscr{H} = \operatorname{Log}\beta$ is the topological entropy of the β-shift (Proposition 5.1 in [PaPt]), in particular when β tends to 1^+. This extended research of zeroes is general and concerns (i) the meromorphic extension of the dynamical zeta function of a dynamical system outside the disk of convergence whose radius is $\exp(-\mathscr{H})$ with $\mathscr{H}$ the topological entropy, the pressure, etc, of the dynamical system, and their poles in this annular region (Haydn [Hy], Hilgert and Rilke [HR], Parry and Pollicott [PaPt], Pollicott [Pt], Ruelle [Ru7]), (ii) the structure theorems of orthogonal decomposition of the (Perron-Frobenius operators) transfer operators, with the geometry of their isolated eigenvalues (e.g. Theorem 1 in Baladi and Keller [BaK]).

How the zeroes of $f_\beta(z)$ lying in the annular region $1/\beta < |z| < 1$ are correlated to the (Galois-) conjugates of an algebraic integer $1 < \beta \leq \theta_2^{-1}$ will be considered in §5 and §6.

Theorem 4.17. *There exists a dense subset of β's in $(1,\infty)$ for which the Parry Upper function $f_\beta(z)$, defined in $|z| < 1$, has infinitely many zeroes lying in the real interval $(-1, -1+\frac{1}{\beta})$, and having -1 as unique limit point.*

Proof. The function $f_\beta(z)$ is holomorphic and nonzero in the open unit disk. As meromorphic function, its domain of definition is $\mathbb{C}$ or $|z| < 1$, depending upon whether β is Parry or not. All the limit points of any subfamilies of zeroes lie necessarily on the boundary $|z| = 1$. The result follows from Theorem 7.1 in [FLP] and Theorem 4.12. The last claim comes from the behaviour of the Parry Upper functions

for β running over the neighbourhoods of the Perron numbers $\theta_n^{-1}, n \geq 3$ (Theorem 4.29). $\square$

The arguments invoked in the proof of Theorem 7.1 in [FLP], used for proving Theorem 4.17, are not compatible with the moderate gappinesses due algebraic numbers β: indeed these authors construct gaps of lengths going extremely fast to infinity, in contradiction with Theorem 4.4 if the algebraicity of β is assumed. Therefore Theorem 4.17 seems to be essentially addressed to transcendental numbers.

Whether Theorem 4.17 is also addressed to an infinite subcollection of algebraic numbers is partially kown: indeed, there exists a "spike" formed by the real negative beta-conjugates of subcollections of simple Parry numbers, on $[-1,0)$, in Solomyak's fractal $\mathscr{G}$ (Theorem 4.18; Figure 1 in [Sk]).

Separating algebraic numbers β from transcendental numbers using lacunarity properties of β-expansions has been considered by several authors (Adamczewski and Bugeaud [AiB], Bugeaud [Bg], Dubickas [Ds12]).

4.2.2 Zero-free regions and density distribution of zeroes in Solomyak's fractal

Let $\beta > 1$ be a real number (algebraic or transcendental). The Parry Upper function $f_\beta(z)$ has its zeroes of modulus < 1 in a region of the open unit disk which is given by Solomyak's constructions in [Sk] §3. Let us recall them, focusing only on the interior of the unit disk (Theorem 4.18). From Theorem 4.12 and (4.19), the zeroes of $f_\beta(z)$ in $|z| < 1$, which are $\neq 1/\beta$, are the zeroes of modulus < 1 of the power series $1 + \sum_{j=1}^{\infty} T_\beta^j(1)\, z^j$. Then let

$$\mathscr{B} := \{h(z) = 1 + \sum_{j=1}^{\infty} a_j z^j \mid a_j \in [0,1]\}$$

be the class of power series defined on $|z| < 1$ equipped with the topology of uniform convergence on compacts sets of $|z| < 1$. The subclass $\mathscr{B}_{0,1}$ of $\mathscr{B}$ denotes functions whose coefficients are all zeros or ones. The space $\mathscr{B}$ is compact and convex. Let

$$\mathscr{G} := \{\lambda \mid |\lambda| < 1, \exists\, h(z) \in \mathscr{B} \text{ such that } h(\lambda) = 0\} \quad \subset D(0,1)$$

be the set of zeroes of the power series belonging to $\mathscr{B}$. The zeroes gather within the unit circle and curves in $|z| < 1$ given in polar coordinates, by [VG2]. The complement $D(0,1) \setminus (\mathscr{G} \cup \{\frac{1}{\beta}\})$ is a zero-free region for $f_\beta(z)$; the domain $D(0,1) \setminus \mathscr{G}$ is star-convex due to the fact that: $h(z) \in \mathscr{B} \Longrightarrow h(z/r) \in \mathscr{B}$, for any $r > 1$ ([Sk], §3), and that $1/\beta$ is the unique root of $f_\beta(z)$ in $(0,1)$. If β is a Parry number, $\mathscr{G}$ contains all the Galois- and beta-conjugates (if any) of β of modulus < 1.

For every $\phi \in (0, 2\pi)$, there exists $\lambda = re^{i\phi} \in \mathscr{G}$; the point of minimal modulus with argument ϕ is denoted $\lambda_\phi = \rho_\phi e^{i\phi} \in \mathscr{G}$, $\rho_\phi < 1$. A function $h \in \mathscr{B}$ is called ϕ-optimal if $h(\lambda_\phi) = 0$. Denote by $\mathscr{K}$ the subset of $(0,\pi)$ for which there exists a ϕ-optimal function belonging to $\mathscr{B}_{0,1}$. Denote by $\partial\mathscr{G}_S$ the "spike": $[-1, \frac{1}{2}(1-\sqrt{5})]$ on the negative real axis.

Theorem 4.18 (Solomyak). *(i) The union $\mathcal{G} \cup \mathbb{T} \cup \partial\mathcal{G}_S$ is closed, symmetrical with respect to the real axis, has a cusp at $z = 1$ with logarithmic tangency (Figure 1 in* [Sk]*),*

(ii) the boundary $\partial\mathcal{G}$ is a continuous curve, given by $\phi \to |\lambda_\phi|$ on $[0,\pi)$, taking its values in $[\frac{\sqrt{5}-1}{2}, 1)$, with $|\lambda_\phi| = 1$ if and only if $\phi = 0$. It admits a left-limit at π^-, $1 > \lim_{\phi\to\pi^-} |\lambda_\phi| > |\lambda_\pi| = \frac{1}{2}(-1+\sqrt{5})$, the left-discontinuity at π corresponding to the extremity of $\partial\mathcal{G}_S$.

(iii) at all points $\rho_\phi e^{i\phi} \in \mathcal{G}$ such that ϕ/π is rational in an open dense subset of $(0,2)$, $\partial\mathcal{G}$ is non-smooth,

(iv) there exists a nonempty subset of transcendental numbers L_{tr}, of Hausdorff dimension zero, such that $\phi \in (0,\pi)$ and $\phi \notin \mathcal{K} \cup \pi\mathbb{Q} \cup \pi L_{tr}$ implies that the boundary curve $\partial\mathcal{G}$ has a tangent at $\rho_\phi e^{i\phi}$ (smooth point).

Proof. [Sk], § 3 and § 4. □

Solomyak's (semi-)fractal $\mathcal{G}$ contains the set $\overline{W}$, where W consists of zeroes λ, $|\lambda| < 1$, of polynomials $1 + \sum_{j=1}^{q} a_j z^j$ having all coefficients a_j zeroes and ones, studied by Odlyzko and Poonen [OP]. Inside $\mathcal{G}$ the density distribution of the zeroes of $f_\beta(z)$ admits a majorant function, as follows.

Theorem 4.19. *For all $\beta > 1$, $f_\beta(z)$ has no zero in $|z| < \frac{1}{\beta}$ and all its zeroes $\neq 1/\beta$, of modulus < 1, lie in the interior $\overset{o}{\mathcal{G}}$. For all $\beta \in (1, \theta_6^{-1})$ and r satisfying $\frac{1}{\beta} \leq r < 1$, $f_\beta(z)$ has*

$$\leq \; c_{(\beta)} \times \frac{\operatorname{Log}(1-r)}{\operatorname{Log} r} \tag{4.26}$$

zeroes in the annular region $\frac{1}{\beta} \leq |z| \leq r$, with

$$c_{(\beta)} = \frac{\operatorname{Log}(1-\frac{1}{\beta}) - \operatorname{Log}(2-\frac{1}{\beta})}{(\frac{\beta-1}{\operatorname{Log}(\beta-1)}+1)\operatorname{Log}(1-\frac{1}{\beta})}.$$

Proof. The number of zeroes in $|z| < 1/\beta$ is equal to 0 [FLP]. The coefficients $T_\beta^j(1)$, $j \geq 1$, of $1 + \sum_{j=1}^{\infty} T_\beta^j(1) z^j$ are all fractional parts, then $\neq 1$. If $f_\beta(z)$ admits a zero on the boundary $\partial\mathcal{G}$, then $1 + \sum_{j=1}^{\infty} T_\beta^j(1) z^j$ would be ϕ-optimal for some $\phi \in (0,\pi]$. By Lemma 3.4 in [Sk], at least one coefficient $T_\beta^j(1)$ should be equal to 1. Contradiction.

The density distribution of zeroes relies upon Jensen's formula applied to the circle $|z| = R$ with $r < R < 1$. For all $\beta > 1$, the constant term is $f_\beta(0) = -1$. Let $\beta \in (\theta_{n+1}^{-1}, \theta_n^{-1})$, $n \geq 1$. Here R will be taken equal to $r^{1/n}$. If $z_1, z_2, \ldots, z_m$ are the

zeroes of $f_\beta(z)$ in $|z| < R$, and Q the number of zeroes in $|z| < r$, then

$$\sum_{q=1}^{m} \mathrm{Log}\,(\frac{R}{|z_j|}) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Log}\,|f_\beta(Re^{it})|dt.$$

Therefore,

$$Q(\mathrm{Log}\,R - \mathrm{Log}\,r) \leq \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Log}\,|f_\beta(Re^{it})|dt.$$

Since $n+1 = \mathrm{dyg}(\beta)$, by Theorem 4.6, $d_\beta(1) = 0.10^{n-1}10^{n_1}10^{n_2}1\ldots$, with $n_j \geq n-1$ for $j \geq 1$, we deduce

$$|f_\beta(Re^{it})| \leq \sum_{j=0}^{\infty} R^j \leq 1 + R + R^{n+1}\sum_{q=0}^{\infty} R^{nq} \leq 2 + \frac{r}{1-r},$$

$$\text{and} \qquad Q \;\leq\; \frac{\mathrm{Log}\,(\frac{2-r}{1-r})}{(\frac{1}{n}-1)\,\mathrm{Log}\,r}.$$

By Proposition 5.5, for $n \geq 6$, $n = -\frac{\mathrm{Log}\,(\beta-1)}{\beta-1} + \ldots$, whose proof is obtained independently. Therefore

$$Q \;<\; \frac{\mathrm{Log}\,(1-r) - \mathrm{Log}\,(2-r)}{(\frac{\beta-1}{\mathrm{Log}\,(\beta-1)}+1)\,\mathrm{Log}\,r}. \tag{4.27}$$

For any $1 < \beta \leq \theta_6^{-1}$, the function:

$$r \to \frac{-\mathrm{Log}\,(2-r) + \mathrm{Log}\,(1-r)}{(\frac{\beta-1}{\mathrm{Log}\,(\beta-1))}+1)\,\mathrm{Log}\,(1-r)} \qquad \text{is decreasing on } \;[1/\beta, 1)$$

and takes its maximum at $r = 1/\beta$. We deduce the claim. $\square$

In § 5.3 and § 6.2 we will show much more, i.e. the existence of lenticuli of roots of the Parry Upper functions $f_\beta(z)$, for $\beta > 1$ small enough, in $\mathscr{G}$, spreading inside the cusp region of $\partial\mathscr{G}$, stemming from $1/\beta$, in the neighbourhood of $z = 1$ towards $e^{\pm i\frac{\pi}{3}}$. The minoration of the Mahler measure $\mathrm{M}(\beta)$, for $\beta > 1$ being an algebraic integer, will be deduced from the explicit asymptotic expansions of these lenticular roots.

The identification of zeroes of analytical functions, which lie very close to natural boundaries, is a difficult problem. For power series having Hadamard lacunarity Fuchs [Fs] obtained results. Here the power series have coefficients in a finite set with moderate lacunarity, making the problem more delicate, though general theorems exist (Levinson [Lsn] Chap. VI, Robinson [Ron]).

4.3 Dichotomy of Perron numbers, Pisot numbers, Salem numbers, Blanchard's classification, Parry numbers, Ito-Sadahiro (Irrap) numbers, negative β-shift

The set $\mathbb{P}$ of Perron numbers contains the subset $\mathbb{P}_P$ of all (simple and nonsimple) Parry numbers by a result of Lind [Ld] (Blanchard [Bd], Boyle [Ble], Denker, Grillenberger and Sigmund [DGS], Frougny in [Lo] chap.7). The set $\mathbb{P} \setminus \mathbb{P}_P$ is not empty, at least by Akiyama's Theorem 4.20; it would contain all Salem numbers of large degrees by Thurston [Tn2] p. 11. Parry ([Pa], Theorem 5) proved that the subcollection of simple Parry numbers is dense in $[1,\infty)$. Simple Parry numbers β often satisfy the minoration (§ 4.7): $\mathrm{M}(\beta) \geq \Theta$. In the opposite direction a Conjecture of K. Schmidt [Sdt] asserts that Salem numbers are all Parry numbers. Boyd [Bo15] established a simple probabilistic model, based on the frequencies of digits occurring in the Rényi β-expansions of unity, to conjecture that, more realistically, Salem numbers are dispatched into the two sets of Parry numbers and nonParry numbers, each of them with densities > 0. This dichotomy of Salem numbers was reconsidered by Hichri [Hi] [Hi2] [Hi3]. Few examples of nonParry algebraic numbers > 1 exist; Solomyak ([Sk] p. 483) gives $\frac{1}{2}(1+\sqrt{13})$.

Theorem 4.20 (Akiyama). *The dominant root $\gamma_n > 1$ of $-1-z+z^n$, for $n \geq 2$, is a Perron number which is a Parry number if and only if $n = 2,3$. If $n = 2,3$, $\gamma_2 = \theta_2^{-1}$ and $\gamma_3 = \theta_5^{-1} = \Theta$ are Pisot numbers which are simple Parry numbers.*

Proof. Theorem 1.1 and Lemma 2.2 in [Aa] using Lagrange inversion formula. The dynamics of the Perron numbers θ_n^{-1} for $n \geq 2$ is reported in § 4.5. □

This dichotomy separates the set of real algebraic integers > 1 into two disjoint nonempty parts; in particular, in restriction, the set of Salem numbers, as: $\mathrm{T} \cap \mathbb{P}_P$ vs. $\mathrm{T} \setminus \mathrm{T} \cap \mathbb{P}_P$. The small Salem numbers found by Lehmer in [Le], reported below, either given by their minimal polynomial or equivalently by their β-expansion ("dynamization"), are Parry numbers:

$\deg(\beta)$	$\beta = \mathrm{M}(\beta)$	minimal pol. of β	$d_\beta(1)$
4	$1.722\ldots$	$X^4-X^3-X^2-X+1$	$0.1(100)^\omega$
6	$1.401\ldots$	$X^6-X^4-X^3-X^2+1$	$0.1(0^2 10^4)^\omega$
8	$1.2806\ldots$	$X^8-X^5-X^4-X^3+1$	$0.1(0^5 10^5 10^7)^\omega$
10	$1.17628\ldots$	$X^{10}+X^9-X^7-X^6-X^5$ $-X^4-X^3+X+1$	$0.1(0^{10}10^{18}10^{12}10^{18}10^{12})^\omega$

The respective dynamical degrees $\mathrm{dyg}(\beta)$ of the last two Salem numbers β are 7 and 12, with Parry polynomials of respective degrees 20 and 75, given by (4.24), for which $f_\beta(z) =$

$$-\frac{z^{20}-z^{19}-z^{13}-z^7-z+1}{1-z^{19}} = -1+z+z^7+z^{13}+\ldots = G_7(z)+z^{13}+\ldots, \quad (4.28)$$

resp.

$$f_\beta(z) = -\frac{z^{75} - z^{74} - z^{63} - z^{44} - z^{31} - z^{12} - z + 1}{1 - z^{74}}$$

$$= -1 + z + z^{12} + z^{31} + \ldots = G_{12}(z) + z^{31} + \ldots. \quad (4.29)$$

Proposition 4.21. *Let $\beta > 1$ be an algebraic number which admits k real (Galois) conjugates distinct of β and are > 0. If $k = 1$, then β is neither a nonsimple Parry number for which $d_\beta(1)$ is purely periodic, nor a simple Parry number. If $k \geq 2$, then β is not a Parry number.*

Proof. Let $k = 1$. If β is a simple Parry number, then $d_\beta(1) = 0.t_1t_2\ldots t_N$ would denote the Rényi β-expansion of 1, for some integer $N \geq 2$, with $t_N \neq 0$. Thus β would be a root of the polynomial $X^N - t_1X^{N-1} - t_2X^{N-2} - \ldots - t_N$. Since all the coefficients $-t_j$ are ≤ 0, the number of changes of sign in the coefficient vector of this polynomial is equal to 1. By Descartes's rule of signs this polynomial would possess only one positive real root. Contradiction. The proof is the same if we assume the pure periodicity ((4.25) admits one change of sign).

Let $k \geq 2$. If β is assumed a Parry number, simple or not, the number of sign changes in the Parry polynomial is either equal to 1 or 2, according to the type of Parry polynomial (4.23) (4.25) or (4.24). The number of positive real roots would be ≤ 2. Contradiction. □

A real algebraic integer $\beta > 1$ close to one is commonly given by its minimal polynomial, not by its β-expansion of unity, though it is equivalent. The Parry Upper function at β is a sparse power series having a coefficient vector with gaps of zeroes (missing monomials) of minimal length controlled by the dynamical degree $\mathrm{dyg}(\beta)$; to examplify the role played by $\mathrm{dyg}(\beta)$, we will reverse the representation of the smallest Salem numbers $\beta > 1$ in the following, by considering their β-expansions.

The smallest Salem numbers of degree ≤ 44 are all known from the complete list of Mahler measures ≤ 1.3 of Mossinghoff [Mlist] of irreducible monic integer polynomials of degree ≤ 180. Table 1 gives the subcollection of those Salem numbers β which are Parry numbers within the intervals of extremities the Perron numbers $\theta_n^{-1}, n = 5, 6, \ldots, 12$. In each interval $\mathrm{dyg}(\beta)$ is constant while the increasing order of the βs corresponds to a certain disparity of the degrees $\deg(\beta)$. The remaining Salem numbers in [Mlist] are very probably nonParry numbers though proofs are not available yet; they are not included in Table 1. Apart from them, the other Salem numbers which exist in the intervals $(\theta_n^{-1}, \theta_{n-1}^{-1}), n \geq 6$, if any, should be of (usual) degrees $\deg > 180$.

dyg	deg	β	$P_{\beta,P}$	$d_\beta(1)$
5	3	$\theta_5^{-1} = 1.324717$	5	$0.10^3 1$
6	18	1.29567	22	$0.1(0^4 10^9 10^6)^\omega$
6	10	1.293485	12	$0.1(0^4 10^6)^\omega$
6	24	1.291741	24 irr.	$0.1(0^4 10^{11} 10^6)^\omega$
6	26	1.286730	30	$0.1(0^4 10^{17} 10^6)^\omega$
6	34	1.285409	38	$0.1(0^4 10^{25} 10^6)^\omega$
6	30	1.285235	45	$0.1(0^4 10^{32} 10^6)^\omega$
6	44	1.285199	66	$0.1(0^4 10^{54} 10^6)^\omega$
6	6	$\theta_6^{-1} = 1.285199$	6 irr.	$0.10^4 1$
7	26	1.285196	44	$0.1(0^5 10^5 10^5 10^5 10^5 10^5 10^7)^\omega$
7	26	1.281691	..	$0.1(0^5 10^5 10^9 10^5 10^{17} 10^7 10^6 10^6 10^7 10^{12})^\omega$
7	8	1.280638	20	$0.1(0^5 10^5 10^7)^\omega$
7	10	1.261230	14	$0.1(0^5 10^7)^\omega$
7	24	1.260103	28	$0.1(0^5 10^{13} 10^7)^\omega$
7	18	1.256221	36	$0.1(0^5 10^{21} 10^7)^\omega$
7	7	$\theta_7^{-1} = 1.255422$	7 irr.	$0.10^5 1$
8	18	1.252775	120	$0.1(0^6 10^6 10^{10} 10^{16} 10^{12} 10^7 10^{12} 0^{16} 10^{10} 10^6 10^8)^\omega$
8	12	1.240726	48	$0.1(0^6 10^{11} 10^7 10^{11} 10^8)^\omega$
8	20	1.232613	41	$0.1(0^6 10^{24} 10^8)^\omega$
8	8	$\theta_8^{-1} = 1.232054$	8 irr.	$0.10^6 1$
9	10	1.216391	18	$0.1(0^7 10^9)^\omega$
9	9	$\theta_9^{-1} = 1.213149$	9 irr.	$0.10^7 1$
10	14	1.200026	20	$0.1(0^8 10^{10})^\omega$
10	10	$\theta_{10}^{-1} = 1.197491$	10 irr.	$0.10^8 1$
11	9	$\theta_{11}^{-1} = 1.184276$	11	$0.10^9 1$
12	10	1.176280	75	Lehmer's number : $0.1(0^{10} 10^{18} 10^{12} 10^{18} 10^{12})^\omega$
12	12	$\theta_{12}^{-1} = 1.172950$	12 irr.	$0.10^{10} 1$

Table 1. Smallest Salem numbers $\beta < 1.3$ of degree ≤ 44, which are Parry numbers, from [Mlist]. In Column 1 is reported the dynamical degree of β (cf Theorem 1.1 for its asymptotic expression). Column 4 gives the degree of the Parry polynomial $P_{\beta,P}$ of β; $P_{\beta,P}$ is reducible except if "irr." is mentioned.

Generalizing (4.28) and (4.29) to the Salem numbers β and Perron numbers of Table 1, the Parry Upper function at β takes the general form, after Theorem 4.6:

$$f_\beta(z) = G_{\mathrm{dyg}(\beta)} + z^{n_1} + \ldots \qquad \text{with } n_1 \geq 2\,\mathrm{dyg}(\beta) - 1. \tag{4.30}$$

For some families of algebraic integers, this "dynamization" of the minimal polynomial is known explicitly, the digits being algebraic functions of the coefficients of the minimal polynomials: e.g. for Salem numbers of degree 4 and 6 (Boyd [Bo11] [Bo12]), for Salem numbers of degree 8 (Hichri [Hi] [Hi2] [Hi3]), for Pisot numbers (Boyd [Bo15], Frougny and Solomyak [FyS], Bassino [Bso] in the cubic case, Hare [H], Panju [Pu] for regular Pisot numbers). Schmidt [Sdt], independently Bertrand-Mathis [BMs], proved that Pisot numbers are Parry numbers. Many Salem numbers are known to be Parry numbers. For Salem numbers of degree 4 it is the case [Bo13]. For Salem numbers of degree ≥ 6, Boyd [Bo14] gave an heuristic argument and a probabilistic model, for the existence of nonParry Salem numbers as a metric approach of the dichotomy of Salem numbers. This approach, coherent with Thurston's one ([Tn2], p. 11), is in contradiction with the conjecture of Schmidt. Hichri [Hi] [Hi2] [Hi3] further developped the heuristic approach of Boyd for Salem numbers of degree 8. The Salem numbers of degree ≤ 8 are all greater than $1.280638\ldots$ from [Mlist].

Using the "Construction of Salem", Hare and Tweedle [HTe] obtain convergent families of Salem numbers, all Parry numbers, having as limit points the limit points of the set S of Pisot numbers in the interval $(1,2)$ (characterized by Amara [Ama]). These families of Parry Salem numbers do not contain Salem numbers smaller than Lehmer's number.

Parry numbers are studied from the negative β-shift. The negative β-shift was introduced by Ito and Sadahiro [IS] (Liao and Steiner [LS], Masakova and Pelantova [MP], Nguéma Ndong [N]) and the generalized β-shift by Gora [Ga] [Ga2] and Thompson [Th]), in the general context of iterated interval maps and post-critical finite (PCF) interval maps [MorT] [Tn2]. Indeed, Kalle [Ke] showed that nonisomorphisms exist between the β-shift and the negative β-shift, possibly leading to new Parry numbers arising from "negative" Parry numbers (called Ito-Sadahiro numbers in [MP], Irrap numbers in [LS] reading "Parry" from the right to the left). More generally negative Parry numbers and generalized Parry numbers are defined as poles of the corresponding dynamical zeta functions [N] [Th2]. Negative Pisot and Salem numbers appear naturally in several domains: as roots of Newman polynomials [HMf], in association equations with negative Salem polynomials [GdVG], in topology with the Alexander polynomials of pretzel links (§ 2), as Coxeter polynomials for Coxeter elements (Hironaka [Ha]; § 2), in studies of numeration with negative bases (Frougny and Lai [FyL]). Generalizing Solomyak's constructions to the generalized β-shift, Thompson [Th2] investigates the fractal domains of existence of the conjugates.

In terms of Parry Upper functions $f_\beta(z)$ the dichotomy between Parry numbers and nonParry numbers (Theorem 4.14) corresponds to a rationality criterium. By (4.19) and (4.20) the structure properties of the Parry Upper function $f_\beta(z)$ and the dynamical zeta function $\zeta_\beta(z)$ (1.25) are based on the knowledge of the number of orbits of 1 under T_β, of length dividing n, $n \to \infty$, and the topological properties of the set $\{T_\beta^n(1)\}$. On this basis Blanchard [Bd] proposed a classification of real numbers

$\beta > 1$ into five classes; Verger-Gaugry in [VG] refined it in terms of asymptotic gappiness in the direction of more enlighting the algebraicity of β:

Class C1: $d_\beta(1)$ is finite,
Class C2: $d_\beta(1)$ is ultimately periodic but not finite,
Class C3: $d_\beta(1)$ contains bounded strings of zeroes, but is not ultimately periodic (0 is not an accumulation point of $\{T_\beta^n(1)\}$),
Class C4: $\{T_\beta^n(1)\}$ is not dense in $[0,1]$, but admits 0 as an accumulation point,
Class C5: $\{T_\beta^n(1)\}$ is dense in $[0,1]$.

Apart from C1, resp. C2, which is exactly the set of simple, resp. nonsimple, Parry numbers, how the remaining algebraic numbers > 1 are dispatched in the classes C3, C4 and C5 is obscure. The specification property, meaning that 0 is not an accumulation point for $\{T_\beta^n(1)\}$, was weakened by Pfister and Sullivan [PS] and Thompson [Th]. Schmeling [Sg] proved that the class C3 has full Hausdorff dimension and that the class C5, probably mostly occupied by transcendental numbers, is of full Lebesgue measure 1. Lacunarity and Diophantine approximation were investigated by Bugeaud and Liao [BgLo], Hu, Tong and Yu [HTY], Li, Persson, Wang and Wu [LPWW]. For any $x_0 \in [0,1]$ the asymptotic distance $\liminf_{n\to\infty} |T_\beta^n(1) - x_0|$, for almost all $\beta > 1$ (for the Lebesgue measure), was studied by Persson and Schmeling [PSg] [Sg], Ban and Li [BL], Cao [Co], Fang, Wu and Li [FWL], Li and Chen [LC], Lü and Wu [LW], Tan and Wang [TW]. Kwon [Kwn] studies the subset of Parry numbers whose conjugates lie close to the unit circle, using technics of combinatorics of words. Adamczewski and Bugeaud ([AiB], Theorem 4) show that the class C4 contains self-lacunary numbers, all transcendental, from Schmidt's Subspace Theorem and results of Corvaja and Zannier.

4.4 Cyclotomic jumps in families of Parry Upper functions, right-continuity

Allowing the real base β to vary continuously in the neighbourhood $[1, \theta_2^{-1})$ of 1, except 1, asks the question whether it has a sense to consider the continuity of the bivariate Parry Upper function $(\beta, z) \to f_\beta(z)$, and, if it is the case, on which subsets, in z, of the complex plane.

Theorem 4.24 and its Corollary show that the open unit disk is a domain where the continuity of the roots of $f_\beta(z)$ in $|z| < 1$ holds though the functions $\mathrm{f}_\beta(z)$ are only right continuous in β, with infinitely many cyclotomic jumps, while, in the complement $|z| \geq 1$, either the Parry Upper functions are not defined, or may exhibit drastic changes on the unit circle.

Lemma 4.22. *Let $1 < \beta < \theta_2^{-1}$ and $0 < x < 1$. Then (i) the bivariate β-transformation map $(\beta, x) \to T_\beta(x) = \{\beta x\} = \beta x - \lfloor \beta x \rfloor$ is continuous, in β and x, when βx is not a*

positive integer. If βx is a positive integer, $x = 1/\beta$ and

$$\lim_{y \to \frac{1}{\beta}^-, \gamma \to \beta^-} T_\gamma(y) = 1, \qquad \lim_{y \to \frac{1}{\beta}^+, \gamma \to \beta^+} T_\gamma(y) = T_\beta(\frac{1}{\beta}) = 0; \tag{4.31}$$

(ii) for any (β, x), there exists $\varepsilon = \varepsilon_{\beta,x}$ such that $T_\gamma(y)$ is increasing both in $\gamma \in [\beta, \beta+\varepsilon)$ and in $y \in [x, x+\varepsilon)$.

Proof. Lemma 3.1 in [FLP]. (i) If βx is an integer, this integer is 1 necessarily. The value $x = 1/\beta$ is a negative power of β. The Rényi β-expansion of $1/\beta$ is deduced from $d_\beta(1)$ by a shift, given in (4.3) and (4.6); the sequence $(T_\beta^n(\frac{1}{\beta}))_{n\geq 1}$ is directly obtained from $(T_\beta^n(1))_{n\geq 1}$. The fractional part $\gamma \to \{\gamma\} = T_\gamma(1)$ is right continuous, hence the result; (ii) obvious. □

Lemma 4.23. *Let $\beta \in (1, \theta_2^{-1})$. (i) If β is a simple Parry number, then, for all $n \geq 1$, the map $\gamma \to T_\gamma^n(1)$ is right continuous at β:*

$$\lim_{\gamma \to \beta^+} T_\gamma^n(1) = T_\beta^n(1), \tag{4.32}$$

(ii) if β is a simple Parry number, such that $T_\beta^N(1) = 0$ with $T_\beta^k(1) \neq 0$, $1 \leq k < N$, then, for all $n \geq 1$,

$$\lim_{\gamma \to \beta^-} T_\gamma^n(1) = \begin{cases} T_\beta^n(1), & n < N \quad (\textit{left continuity}) \\ T_\beta^{n_N}(1), & n \geq N, \end{cases} \tag{4.33}$$

where $n_N \in \{0, 1, \ldots N-1\}$ is the residue of n modulo N,

(iii) if β is a nonsimple Parry number, then $\gamma \to T_\gamma^n(1)$ is continuous at β:

$$\lim_{\gamma \to \beta^-} T_\gamma^n(1) = T_\beta^n(1) = \lim_{\gamma \to \beta^+} T_\gamma^n(1), \qquad \textit{for all } n \geq 1. \tag{4.34}$$

Proof. Lemma 3.2 in [FLP]. □

Denote by

$$\mathscr{F} := \{f_{\beta}{}_{|_{|z|<1}}(z) \mid 1 < \beta < \theta_2^{-1}\}$$

the set of the restrictions of the Parry Upper functions $f_\beta(z)$, $1 < \beta < \theta_2^{-1}$, to the open unit disk. The set $\mathscr{F}$ is equipped with the topology of the uniform convergence on compact subsets of $|z| < 1$.

Theorem 4.24. *In $\mathscr{F}$ the following right and left limits hold: (i) if β be a nonsimple Parry number, then continuity occurs as:*

$$\lim_{\gamma \to \beta^-} f_\gamma(z) = f_\beta(z) = \lim_{\gamma \to \beta^+} f_\gamma(z), \tag{4.35}$$

(ii) if β is a simple Parry number, and N the minimal value for which $T_\beta^N(1) = 0$, then

$$\lim_{\gamma \to \beta^+} f_\gamma(z) = f_\beta(z), \tag{4.36}$$

$$\lim_{\gamma \to \beta^-} f_\gamma(z) = \frac{f_\beta(z)}{(1 - z^N)}. \tag{4.37}$$

Proof. Let $\gamma, \beta \in (1, \theta_2^{-1})$ with $|\gamma - \beta| \le \varepsilon$, $\varepsilon > 0$, $d_\gamma(1) = 0.t'_1 t'_2 \ldots$ and $d_\beta(1) = 0.t_1 t_2 \ldots$. Any compact subset of $|z| < 1$ is included in a closed disk centered at 0 of radius r for some $0 < r < 1$. Assume $|z| \le r$. (i) Assume β nonsimple. Since $|T_\gamma^m(1) - T_\beta^m(1)| \le 2$ for $m \ge 1$, then

$$|f_\gamma(z) - f_\beta(z)| = \Big|\sum_{n \ge 1} (t'_n - t_n) z^n\Big| = \Big|\sum_{n \ge 1} [(\gamma T_\gamma^{n-1}(1) - \beta T_\beta^{n-1}(1)) - (T_\gamma^n(1) - T_\beta^n(1))] z^n\Big|$$

$$\le \sum_{n \ge 1} \Big|(\gamma T_\gamma^{n-1}(1) - \beta T_\beta^{n-1}(1)) - (T_\gamma^n(1) - T_\beta^n(1))\Big| r^n \le 2(\varepsilon + \beta + 1) \sum_{n \ge 1} r^n, \tag{4.38}$$

which is convergent. By (4.34) and the Lebesgue dominated convergence theorem, taking the limit termwise in the summation,

$$\lim_{\gamma \to \beta} |f_\gamma(z) - f_\beta(z)| = 0, \qquad \text{uniformly for } |z| \le r.$$

(ii) By (4.32) and (4.33), the iterates of 1 under the γ-transformation $T_\gamma^n(1)$ behave differently at β if $\gamma < \beta$ or resp. $\gamma > \beta$ when γ tends to β: if $\gamma > \beta$, we apply the Lebesgue dominated convergence theorem in (4.38) to obtain the right continuity at β, i.e. (4.36); if $\gamma \to \beta^-$, (4.37) comes from the dominated convergence theorem applied to

$$f_\gamma(z) - \frac{1}{1 - z^N} f_\beta(z) = (\beta z - 1) \left[\Big(\sum_{n=0}^{\infty} T_\gamma^n(1) \frac{\gamma z - 1}{\beta z - 1} z^n \Big) - \Big(\sum_{q=0}^{\infty} \sum_{m=0}^{N-1} T_\beta^m(1) z^{m+qN} \Big) \right].$$

□

Theorem B in Mori [Mo], on the continuity properties of spectra of Fredholm matrices, admits the following counterpart in terms of the Parry Upper functions:

Corollary 4.25. *The root functions of $f_\beta(z)$ valued in $|z| < 1$ are all continuous, as functions of $\beta \in (1, \theta_2^{-1}) \setminus \bigcup_{n \ge 3} \{\theta_n^{-1}\}$.*

Proof. Let $(\gamma_i)_{i \ge 1}$ be a sequence of real numbers tending to β. The (restrictions, to the open unit disk, of the) functions $f_{\gamma_i}(z)$ constitute a convergent sequence in $\mathscr{F}$, tending either to $f_\beta(z)$ or $f_\beta(z)/(1 - z^N)$ for some integer $N \ge 1$. By Hurwitz's Theorem ([SsZ] (11.1)) any disk in $|z| < 1$, whose closure does not intersect the unit circle, which contains a zero $w(\beta)$ of $f_\beta(z)$ also contains a zero of $f_{\gamma_i}(z)$ for all $i \ge i_0$,

for some i_0. The multiplicity of $w(\beta)$ is equal to the number of zeroes $w(\gamma_i)$, counted with multiplicities, in this disk. □

Theorem 4.26. *(i) If $\beta > 1$ is an algebraic integer for which the Mahler measure $\mathrm{M}(\beta) < \Theta = \theta_5^{-1}$, then β is not a simple Parry number; (ii) conversely, if $1 < \beta < 2$ is a simple Parry number for which the Mahler measure of the complementary factor $\mathrm{M}(\frac{P_{\beta,P}}{P_\beta}) = 1$, equivalently for which the complementary factor $\frac{P_{\beta,P}(X)}{P_\beta(X)}$ is a product of cyclotomic polynomials, then $\mathrm{M}(\beta) = \mathrm{M}(P_\beta) = \mathrm{M}(P_{\beta,P}) \geq \Theta$.*

Proof. By the Theorem of C. Smyth [Sy] the minimal polynomial $P_\beta(X)$ of β is reciprocal. Then β and $1/\beta$ are (Galois-) conjugated and are two real positive roots of P_β. But the Parry polynomial $P_{\beta,P}(X)$ of β is a multiple of $P_\beta(X)$. Therefore the Parry polynomial $P_{\beta,P}$ of β would also have at least two real positive roots. If we assume that β is a simple Parry number, the number of positive real roots of its Parry polynomial should be equal to 1 since $P_{\beta,P}(X)$ admits only one change of signs, by Descartes's rule of signs. From Proposition 4.21 we deduce the contradiction. The converse is obvious from (i) and by Smyth's Theorem [Sy]. □

An example of sequence of simple Parry numbers which are Perron numbers tending to 1^+ is the sequence $(\gamma_{n,k})$ of the dominant roots $\gamma_{n,k}$ of the trinomials $X^n - X^k - 1$ for $1 \leq k < n$ and $n \to +\infty$. Indeed, the result of Flammang [Fg3], who proved the conjecture of C. Smyth for height one trinomials, for n large enough, gives $\lim_{n\to\infty} \mathrm{M}(\gamma_{n,k}) = 1.38135\ldots > \Theta = 1.3247\ldots$ as a true limit, the trinomials $X^n - X^k - 1$ being all Parry polynomials by the self-admissibility of the coefficient vectors (taking care of the factorization is useless). As soon as n is large enough, the inequality $\mathrm{M}(\gamma_{n,k}) \geq \Theta$ is fulfilled.

Another consequence, in $\mathscr{F}$, is the disappearance of the cyclotomic jumps of the left-discontinuities at the algebraic integers $\beta > 1$ close to 1^+, of small measure.

Corollary 4.27. *If $\beta \in (1, \theta_2^{-1})$ is an algebraic integer for which the Mahler measure $\mathrm{M}(\beta) < \Theta = \theta_5^{-1}$, then the left and right continuity of the Parry Upper function occurs in $\mathscr{F}$ as:*

$$\lim_{\gamma\to\beta^-} f_\gamma(z) = f_\beta(z) = \lim_{\gamma\to\beta^+} f_\gamma(z). \tag{4.39}$$

Proof. From Theorem 4.24 and Theorem 4.26, the case (4.37) cannot occur. □

4.5 The Rényi-Parry dynamical systems in Perron number base inherited from the trinomials $-1+X+X^n$, and a perturbation theory of Parry Upper functions respecting the lexicographical ordering in the β-shift

Proposition 4.28. *Let $n \geq 2$. The Perron number θ_n^{-1}, dominant root of the trinomial $G_n^*(X) = -X^n + X^{n-1} + 1$, is a simple Parry number for which $T^j_{\theta_n^{-1}}(1)$ is nonzero for $j = 2, 3, \ldots, n-1$, $T^j_{\theta_n^{-1}}(1) = 0$ for $j \geq n$ with $d_{\theta_n^{-1}}(1) = 0.10^{n-2}1$.*

Proof. The first digit t_1 of $d_{\theta_n^{-1}}(1)$ is obviously $\lfloor \theta_n^{-1} \rfloor = 1$ for all $n \geq 2$. Since the identity $G_n^*(\theta_n^{-1}) = 0 = \theta_n^{-n} - \theta_n^{-n+1} - 1$ holds we deduce $\theta_n^{-n+1}(\theta_n^{-1} - 1) = 1$, therefore

$$T_{\theta_n^{-1}}(1) = \{\theta_n^{-1}\} = \theta_n^{-1} - 1 = \frac{1}{\theta_n^{-n+1}} \in (0,1).$$

Then
$$T^j_{\theta_n^{-1}}(1) = \frac{1}{\theta_n^{-n+j}} > 0 \qquad \text{for } j = 1, 2, \ldots, n-1$$

and
$$T^n_{\theta_n^{-1}}(1) = \{\theta_n^{-1} T^{n-1}_{\theta_n^{-1}}(1)\} = \{\frac{\theta_n^{-1}}{\theta_n^{-1}}\} = 0.$$

Consequently $T^j_{\theta_n^{-1}}(1) = 0$ for all $j > n$. By (4.5) we deduce recursively the values of the integers t_j for $j \geq 2$: $t_2 = 0 = t_3 = \ldots = t_{n-1}$, $t_n = 1$ and $t_j = 0$ as soon as $j > n$. □

If $n \geq 2$, as analytic function, the Parry Upper function $f_{\theta_n^{-1}}(z)$ is equal to the height one trinomial $-1 + z + z^n = G_n(z)$, a polynomial. On the other hand the Parry polynomial of θ_n^{-1} is the reciprocal of G_n, and a multiple of its minimal polynomial: $P_{\theta_n^{-1},P}(z) = -G_n^*(z) = X^n - X^{n-1} - 1$ is irreducible by Proposition 3.1 except if $n \equiv 5 \pmod 6$.

It is tempting to establish a *perturbation theory* based on polynomials to try to answer Lehmer's question when β tends to 1. A priori, as "reference" polynomials, the families (G_n) and (G_n^*) are not good candidates since, by Smyth's theorem, they are not reciprocal and therefore have a Mahler measure $> \Theta$, far from 1. In this direction several apparently better starting points were investigated: (i) families of cyclotomic polynomials (Amoroso [A]), (ii) families given by a parametrization of two-variable polynomials constructed from cyclotomics, having minimality properties for the Mahler measure (Ray [Ry2]), (iii) families of perturbed polynomials with the Zhang-Zagier height [Dhe2], (iv) families of polynomials defined by varying coefficients (Sinclair [Si]) and/or having their roots on the unit circle (Mossinghoff, Pinner and Vaaler [MPV]), (v) families of polynomials with coefficients in the ring of integers of a number field and having their roots on the unit circle (Toledano [To]). Starting from polynomials having a Mahler measure equal to 1, as in (i) to (v), seems natural.

The second direction, which is natural in the context of the β-shift, consists in starting from the family of polynomials $(G_n(z))$, but viewed as set of values of the Parry Upper function $f_\beta(z)$ at all $\beta = \theta_n^{-1}$. The theory of perturbation we are looking for is now deduced, in the case where β is a Parry number, from Proposition 4.5 and Theorem 4.14: for $\theta_n^{-1} \leq \beta < \theta_{n-1}^{-1}$, then $d_{\theta_n^{-1}}(1) = 0.10^{n-2}1 \leq_{lex} d_\beta(1) = 0.t_1t_2\ldots <_{lex} d_{\theta_{n-1}^{-1}}(1) = 0.10^{n-3}1$, with a sequence of digits (t_i) which is either finite or ultimately periodic. The lacunarity in (t_i) is controlled by the integer n. From $d_\beta(1)$ the power series $f_\beta(z)$ is deduced and the reciprocal polynomial $P^*_{\beta,P}(z)$ of the Parry polynomial of β is obtained, together with the minimal polynomial of β which is only one of its irreducible factors, of multiplicity one. Though feasible, to recover the minimal polynomial of β, the theory of factorization of Parry polynomials is a deep question [VG3] (for instance using Bombieri's norms, introduced in [BBEM], instead of Mahler measures). We would obtain: $\mathrm{M}(\beta) \leq \mathrm{M}(P^*_{\beta,P})$. The case where β is an algebraic integer which is not a Parry number is more complicated since it relies upon a theory of divisibility of the integer power series (not eventually periodic, with lacunarity controlled by $\mathrm{dyg}(\beta)$) which are the denominators of the dynamical zeta functions $\zeta_\beta(z)$ by integer monic polynomials.

This theory of perturbation respects the lexicographical ordering in the dynamization of the defining equations. It is adapted to the coding of algebraic numbers in the β-shift, β being the important variable. This theory of perturbation of the function $\beta \to f_\beta(z)$ relies first upon the knowledge of the set of Parry numbers, by Theorem 4.12. Finding a rationality criterion for $\zeta_\beta(z)$ is as difficult as solving the Weil's conjectures (Deligne [Dne], Dwork [Dwk], Kedlaya [Kya], Weil [We]). In the neighbourhood of the sequence (θ_n^{-1}) this theory of perturbation takes the following form.

Theorem 4.29. *There exists a decreasing sequence of positive real numbers $(\varepsilon_n)_{n\geq 3}$, tending to 0, such that, for all $n \geq 3$, the condition $\beta \in (\theta_n^{-1} - \varepsilon_n, \theta_n^{-1} + \varepsilon_n)$ implies that $f_\beta(z)$ has $1 + 2\lfloor \frac{n}{6} \rfloor$ simple zeroes in $|z| < 1$, each zero being obtained from the roots $z_{j,n}$ of modulus < 1 of the trinomials $-1 + z + z^n$, by continuity with β.*

Proof. The roots of $G_n(z)$ are all simple, by Proposition 3.3. If β is close enough to θ_n^{-1} the roots of $f_\beta(z)$ which lie in $|z| < 1$ are all simple by Hurwitz's Theorem (Corollary 4.25 and Theorem 4.24). These roots are obtained by continuity from those of $G_n(z)$. $\square$

Remark 4.30. In Theorem 4.29, β is either a transcendental number or an algebraic number. In both cases, a lenticulus of simple zeroes lies in the angular sector $\arg(z) \in (-\pi/3, +\pi/3)$, as a deformed lenticulus of $\mathscr{L}_{\theta_n^{-1}}$. In the sequel, we will reserve the notation $\mathscr{L}_\beta$ for the zeroes of $f_\beta(z)$ identified as Galois conjugates of β, when $\beta > 1$ is an algebraic integer, zeroes close to those of the lenticulus $\mathscr{L}_{\theta^{-1}_{\mathrm{dyg}(\beta)}}$. The difficulty of the identification of the zeroes will be considered in §5.

4.6 The problem of the identification of the zeroes of $f_\beta(z)$ as conjugates of β

Hypothesis (H): let $\beta > 1$ be an algebraic integer. In this paragraph, let us make the assumption that all the zeroes of $f_\beta(z)$ of modulus < 1 are conjugates of β, and that all the conjugates of β of modulus < 1 are zeroes of $f_\beta(z)$.

This assumption is very probably wrong. The difficulty of the identification, in $|z| < 1$, of the zero-locus of $f_\beta(z)$ with the set of zeroes of the minimal polynomial $P_\beta(z)$ will be partially overcome in § 5.4 and § 6.2; it will lead to a new notion of continuity with the "house", of the minorant M_r of M. Nevertheless, interestingly, assumption (H) would lead to the following claim, where the continuity of M itself would occur locally.

Claim 4.31. Assuming (H) there would exist a sequence $(\varepsilon_n)_{n\geq 6}$ of positive real numbers tending to 0 such that any two successive intervals $(\theta_n^{-1} - \varepsilon_n, \theta_n^{-1} + \varepsilon_n)$ and $(\theta_{n+1}^{-1} - \varepsilon_{n+1}, \theta_{n+1}^{-1} + \varepsilon_{n+1})$ are disjoint, $n \geq 6$, and the Mahler measure

$$\mathrm{M} : \bigcup_{n\geq 6} (\theta_n^{-1} - \varepsilon_n, \theta_n^{-1} + \varepsilon_n) \cap \mathscr{O}_{\overline{\mathbb{Q}}} \;\to\; \mathbb{P}, \quad \beta \to \mathrm{M}(\beta)$$

be continuous and take values $\mathrm{M}(\beta) \geq \Theta$ for any algebraic integer β in this set.

Proof. The existence of the sequence $(\varepsilon_n)_n$ comes from Theorem 4.29. By Theorem 4.29 the number of zeroes of modulus < 1 of $f_\beta(z)$ is the same as that of $f_{\theta_n^{-1}}(z)$ as soon as $\beta > 1$ is close enough to a Perron number θ_n^{-1}, $n \geq 3$. By Corollary 4.25 each zero of $f_\beta(z)$ in $|z| < 1$ is a continuous function of β. Assuming (H) these zeroes are conjugates of β. Then the Mahler measure M is a continuous function on the set of algebraic integers

$$\bigcup_{n\geq 3} (\theta_n^{-1} - \varepsilon_n, \theta_n^{-1} + \varepsilon_n) \;\cap\; \mathscr{O}_{\overline{\mathbb{Q}}}. \tag{4.40}$$

Since $\lim_{n\to\infty} \mathrm{M}(\theta_n^{-1}) = \Lambda = 1.38135\ldots > \Theta = 1.3247\ldots$ by Theorem 3.12, and that $\mathrm{M}(\theta_n^{-1}) > \Theta$ for $n \geq 6$ by Proposition 3.14, it is possible to choose all $\varepsilon_n > 0$, for $n \geq 6$, small enough to have $\mathrm{M}(\beta) \geq \Theta$ for any β belonging to the set (4.40). □

4.7 The minoration of $\mathrm{M}(\beta)$ for β a simple Parry number

Theorem 4.32. *If $1 < \beta < 2$ is a simple Parry number for which the complementary factor $\frac{P_{\beta,P}}{P_\beta}$ in the Parry polynomial of β is a product of cyclotomic polynomials, then $\mathrm{M}(\beta) \geq \Theta = \theta_5^{-1} = 1.3247\ldots$.*

Proof. It is a consequence of Proposition 4.21 and Theorem 4.26. The equality $\mathrm{M}(P_\beta) = \mathrm{M}(P_{\beta,P})$ means that the complementary factor $\frac{P_{\beta,P}(X)}{P_\beta(X)}$ admits a Mahler mea-

sure equal to 1, equivalently, since it is monic, that all its roots are roots of unity by Kronecker's theorem. □

How often is the complementary factor a product of cyclotomic polynomials? A partial answer can be given by a variant (Conjecture 16 below) of the Conjecture of Odlyzko and Poonen [OP] when $1 < \beta < 2$ is simple and such that $P_{\beta,P}(X)$ is irreducible, i.e. when the complementary factor is trivial. In this case,

$$\{\beta \in (1,2) \mid \beta \text{ simple Parry number, } P_{\beta,P}(X) = P_\beta(X)\} \qquad \text{would be dense.}$$

Let us recall the two Conjectures.

Conjecture 15 (Odlyzko - Poonen). *Let $\mathscr{P}_{d,+1}$ denote the set of all polynomials of degree d with constant term 1 and with coefficients in $\{0,1\}$. Denote*

$$\mathscr{P}_+ = \bigcup_{d\geq 1} \mathscr{P}_{d,+1}.$$

Then, in $\mathscr{P}_+$, almost all polynomials are irreducible; more precisely, if $\mathscr{I}_{d,+}$ denotes the number of irreducible polynomials in $\mathscr{P}_{d,+1}$, then

$$\lim_{d\to\infty} \frac{\mathscr{I}_{d,+}}{2^{d-1}} = 1.$$

The best account of Conjecture 15 is given by Konyagin (1999): $\mathscr{I}_{d,+1} \gg \frac{2^d}{\operatorname{Log} d}$. Now, changing the last coefficient to -1 gives the following variant.

Conjecture 16. *Let $\mathscr{P}_{d,-1}$ denote the set of all polynomials of degree d with constant term -1 and with coefficients in $\{0,1\}$. Denote $\mathscr{P}_- = \bigcup_{d\geq 1} \mathscr{P}_{d,-1}$. Then, in $\mathscr{P}_-$, almost all polynomials are irreducible; more precisely, if $\mathscr{I}_{d,-}$ denotes the number of irreducible polynomials in $\mathscr{P}_{d,-1}$, then*

$$\lim_{d\to\infty} \frac{\mathscr{I}_{d,-}}{2^{d-1}} = 1.$$

Indeed, for any simple Parry number $\beta < 2$, the opposite $-P^*_{\beta,P}(X)$ of the reciprocal of the Parry polynomial of β has coefficients in $\{0,1\}$ except the constant term equal to -1, as (from (4.23)): $-P^*_{\beta,P}(X) = -1 + t_1 X + t_2 X^2 + \ldots + t_m X^m$ to which Conjecture 16 applies. The irreducibility of $P^*_{\beta,P}(X)$ is equivalent to that of $P_{\beta,P}(X)$.

Consequently, assuming Conjecture 16, the minoration of the Mahler measure $\mathrm{M}(\beta)$, with $1 < \beta < 2$ any algebraic integer, by the smallest Pisot number Θ, appears as a common rule occurring almost everywhere. Further studies on the crucial problem of the irreducibility of integer polynomials with coefficients in a finite set were carried out by Borwein, Erdélyi and Littmann [BEL], Chern [Crn], Dubickas [Ds4], Dubickas [Ds10].

5 Asymptotic expansion of the lenticular minorant of the Mahler measure $\mathrm{M}(\beta)$ for $\beta > 1$ a real algebraic integer close to one

5.1 Asymptotic expansions of a real number $\beta > 1$ close to one and of the dynamical degree $\mathrm{dyg}(\beta)$

Lemma 5.1. *Let $n \geq 6$. The difference $\theta_n - \theta_{n-1} > 0$ admits the following asymptotic expansion, reduced to its terminant:*

$$\theta_n - \theta_{n-1} \;=\; \frac{1}{n} O\left(\left(\frac{\mathrm{Log}\,\mathrm{Log}\, n}{\mathrm{Log}\, n}\right)^2\right), \tag{5.1}$$

with the constant 1 *involved in* $O(\ )$.

Proof. From (3.8) and Lemma 3.5, we have

$$\theta_n = 1 - \frac{\mathrm{Log}\, n}{n}(1-\lambda_n) + \frac{1}{n} O\left(\left(\frac{\mathrm{Log}\,\mathrm{Log}\, n}{\mathrm{Log}\, n}\right)^2\right)$$

with the constant $1/2$ involved in $O(\ )$, and

$$\lambda_n = \frac{\mathrm{Log}\,\mathrm{Log}\, n}{\mathrm{Log}\, n}\left(\frac{1}{1+\frac{1}{\mathrm{Log}\, n}}\right) + O\left(\frac{\mathrm{Log}\,\mathrm{Log}\, n}{n}\right)$$

with the constant 1 in the Big O. Then we deduce

$$\mathrm{D}(\theta_n) - \mathrm{D}(\theta_{n-1}) = \frac{\mathrm{Log}\, n}{n^2} + O\left(\frac{\mathrm{Log}\,\mathrm{Log}\, n}{n^2}\right).$$

The real function $x^{-2}\mathrm{Log}\, x$ on $(1,+\infty)$ is decreasing for $x \geq \sqrt{e}$. Hence the sequence $(\mathrm{D}(\theta_n) - \mathrm{D}(\theta_{n-1}))$ is decreasing for n large enough. By Proposition 3.2 $(\theta_n - \theta_{n-1})_n$ is already known to tend to 0.

Since $\mathrm{tl}(\theta_n) = \frac{1}{n} O\left(\left(\frac{\mathrm{Log}\,\mathrm{Log}\, n}{\mathrm{Log}\, n}\right)^2\right)$, we have

$$\begin{aligned}
\theta_n - \theta_{n-1} &= (\theta_n - \mathrm{D}(\theta_n)) + [\mathrm{D}(\theta_n) - \mathrm{D}(\theta_{n-1})] - (\theta_{n-1} - \mathrm{D}(\theta_{n-1})) \\
&= \mathrm{tl}(\theta_n) + \left(\frac{\mathrm{Log}\, n}{n^2} + O\left(\frac{\mathrm{Log}\,\mathrm{Log}\, n}{n^2}\right)\right) - \mathrm{tl}(\theta_{n-1}) \\
&= \frac{1}{n} O\left(\left(\frac{\mathrm{Log}\,\mathrm{Log}\, n}{\mathrm{Log}\, n}\right)^2\right)
\end{aligned} \tag{5.2}$$

where the constant involved in $O(\ )$ is now $1 = 1/2 + 1/2$. Hence the claim. $\square$

Theorem 5.2. *Let* $n \geq 6$. *Let* $\beta > 1$ *be a real number of dynamical degree* $\mathrm{dyg}(\beta) = n$. *Then* β *can be expressed as:* $\beta = \mathrm{D}(\beta) + \mathrm{tl}(\beta)$ *with* $\mathrm{D}(\beta) = 1+$

$$\frac{\operatorname{Log} n}{n}\left(1-\left(\frac{n-\operatorname{Log} n}{n\operatorname{Log} n+n-\operatorname{Log} n}\right)\left(\operatorname{Log}\operatorname{Log} n - n\operatorname{Log}\left(1-\frac{\operatorname{Log} n}{n}\right)-\operatorname{Log} n\right)\right) \tag{5.3}$$

and

$$\mathrm{tl}(\beta) \;=\; \frac{1}{n} O\left(\left(\frac{\operatorname{Log}\operatorname{Log} n}{\operatorname{Log} n}\right)^2\right), \tag{5.4}$$

with the constant 1 *involved in* $O(\ )$.

Proof. By definition $\theta_n \leq \beta^{-1} < \theta_{n-1}$. The development term of β^{-1} is $D(\beta^{-1}) = D(\beta^{-1} - \theta_n) + D(\theta_n)$, with $|\beta^{-1} - \theta_n| < \theta_n - \theta_{n-1}$. By Lemma 5.1, $D(\theta_n - \theta_{n-1}) = 0$. Therefore $\beta = D(\beta) + \mathrm{tl}(\beta)$ is deduced from $D(\theta_n)$ in (3.8) and from $\beta^{-1} = D(\beta^{-1}) + \mathrm{tl}(\beta^{-1})$ with $-D(\beta^{-1}) + 1 = D(\beta) - 1$ given by (5.3) and $\mathrm{tl}(\beta^{-1}) = \mathrm{tl}(\beta)$ given by (5.4). □

Theorem 5.3. *Let* $\beta \in (1, \theta_6^{-1})$ *be a real number. The asymptotic expansion of the locally constant function* $n = \mathrm{dyg}(\beta)$*, as a function of the variable* $\beta - 1$*, is*

$$n = -\frac{\operatorname{Log}(\beta-1)}{\beta-1}\left[1+O\left(\left(\frac{\operatorname{Log}(-\operatorname{Log}(\beta-1))}{\operatorname{Log}(\beta-1)}\right)^2\right)\right] \tag{5.5}$$

with the constant 1 in $O(\ )$.

Proof. Inverting (5.3) gives the asymptotic expansion of n as a function of β: from (5.3) readily comes

$$n = \frac{\beta}{\beta-1}\operatorname{Log}\left(\frac{\beta}{\beta-1}\right)\left[1+O\left(\left(\frac{\operatorname{Log}\operatorname{Log}\left(\frac{\beta}{\beta-1}\right)}{\operatorname{Log}\left(\frac{\beta}{\beta-1}\right)}\right)^2\right)\right] \tag{5.6}$$

then (5.5) as $\beta \to 1$. □

Remark 5.4. If β runs over the set of Perron numbers $\theta_n^{-1}, n = 5, 6, \ldots, 12$, and over the smallest Parry - Salem numbers $\beta \leq 1.240726\ldots$ of Table 1, the dynamical degree of β (Table 1, Column 1) is the integer part of $D(n)$ in (5.6):

$$\mathrm{dyg}(\beta) = \lfloor \frac{\beta}{\beta-1}\operatorname{Log}\left(\frac{\beta}{\beta-1}\right)\rfloor. \tag{5.7}$$

5.2 Fracturability of the minimal polynomial by the Parry Upper function

In this subsection $\beta > 1$ is assumed to be an algebraic integer.

When $\beta > 1$ is an algebraic integer, it admits a Parry Upper function $f_\beta(z)$ and a minimal polynomial $P_\beta(z)$. In the following the interplay between both analytical functions are investigated when $\beta > 1$ is close to one.

Theorem 5.5. *Let $\beta > 1$ be an algebraic integer such that $\mathrm{M}(\beta) < \Theta$. The following formal decomposition of the (monic) minimal polynomial*

$$P_\beta(X) = P^*_\beta(X) = U_\beta(X) \times f_\beta(X), \tag{5.8}$$

holds, as a product of the Parry Upper function

$$f_\beta(X) = G_{\mathrm{dyg}(\beta)}(X) + X^{m_1} + X^{m_2} + X^{m_3} + \dots. \tag{5.9}$$

with $m_0 := \mathrm{dyg}(\beta)$, $m_{q+1} - m_q \geq \mathrm{dyg}(\beta) - 1$ for $q \geq 0$, and the invertible formal series $U_\beta(X) \in \mathbb{Z}[[X]]$, quotient of P_β by f_β. The specialization $X \to z$ of the formal variable to the complex variable leads to the identity between analytic functions, obeying the Carlson-Polya dichotomy:

$$P_\beta(z) = U_\beta(z) \times f_\beta(z) \quad \begin{cases} \text{on } \mathbb{C} & \text{if } \beta \text{ is a Parry number, with} \\ & U_\beta \text{ and } f_\beta \text{ both meromorphic}, \\ & \\ \text{on } |z| < 1 & \text{if } \beta \text{ is a nonParry number, with } |z| = 1 \\ & \text{as natural boundary for both } U_\beta \text{ and } f_\beta. \end{cases} \tag{5.10}$$

In both cases, the domain of holomorphy of the function $U_\beta(z)$ contains the open disc $D(0, \theta_{\mathrm{dyg}(\beta)-1})$.

Proof. The algebraic integer β lies between two successive Perron numbers of the family $(\theta_n^{-1})_{n\geq 5}$, as $\theta_n^{-1} \leq \beta < \theta_{n-1}^{-1}$, $\mathrm{dyg}(\beta) = n \geq 6$. By Proposition 4.6, the Parry Upper function $f_\beta(z)$ at β has the form (5.9). The algebraic integer β is a Parry number or a nonParry number. In both cases, $f_\beta(\beta^{-1}) = 0$. If $f_\beta(z) = -1 + \sum_{j\geq 1} t_j z^j$, the zero β^{-1} of $f_\beta(z)$ is simple since the derivative of $f_\beta(z)$ satisfies $f'_\beta(\beta^{-1}) = \sum_{j\geq 1} j t_j \beta^{-j+1} > 0$. The other zeroes of $f_\beta(z)$ of modulus < 1 lie in $1/\beta \leq |z| < 1$. Therefore the poles, if any, of $U_\beta(z) = P_\beta(z)/f_\beta(z)$ of modulus < 1 all lie in the annular region $\theta_{\mathrm{dyg}(\beta)-1} < |z| < 1$.

The formal decomposition (5.8), in $\mathbb{Z}[[X]]$, is always possible. Indeed, if we put $U_\beta(X) = -1 + \sum_{j\geq 1} b_j X^j$, and $P_\beta(X) = 1 + a_1 X + a_2 X^2 + \dots a_{d-1} X^{d-1} + X^d$, (with $a_j = a_{d-j}$), the formal identity $P_\beta(X) = U_\beta(X) \times f_\beta(X)$ leads to the existence of the coefficient vector $(b_j)_{j\geq 1}$ of $U_\beta(X)$, as a function of $(t_j)_{j\geq 1}$ and $(a_i)_{i=1,\dots,d-1}$, as: $b_1 = -(a_1 + t_1)$, and, for $r = 2, \dots, d-1$,

$$b_r = -(t_r + a_r - \sum_{j=1}^{r-1} b_j t_{r-j}) \quad \text{with} \quad b_d = -(t_d + 1 - \sum_{j=1}^{d-1} b_j t_{r-j}), \tag{5.11}$$

$$b_r = -t_r + \sum_{j=1}^{r-1} b_j t_{r-j} \quad \text{for } r > d. \tag{5.12}$$

Then $b_j \in \mathbb{Z}$, $j \geq 1$; the integers $b_r, r > d$, are determined recursively by (5.12) by the sequence (t_i) and from the finite subset $\{b_0 = -1, b_1, b_2, \ldots, b_d\}$, itself determined from $P_\beta(X)$ using (5.11). They inherit the asymptotic properties of the asymptotic lacunarity of (t_i) when r is very large [VG]. If R_β denotes the radius of convergence of $U_\beta(z)$ the inequality $R_\beta \geq \theta_{\mathrm{dyg}(\beta)-1}$ can be directly obtained using Hadamard's formula $R_\beta^{-1} = \limsup_{r\to\infty} |b_r|^{1/r}$ and the following Lemma 5.6 (in which $n = \mathrm{dyg}(\beta)$) whose proof is immediate.

Lemma 5.6. *Let* $\varepsilon > 0$ *such that* $\theta_{n-1}^{-1} < \exp(\varepsilon)$. *There exists a constant* $C = C(\varepsilon) \geq \max\{1, \exp(\varepsilon(n-1)) - 1\}$ *such that:*

$$|b_r| \leq C \times \exp(\varepsilon r), \qquad \textit{for all } r \geq 0. \tag{5.13}$$

$\square$

As an example, let us consider $\beta = 1.291741\ldots$ the Salem number of degree 24 given in Table 1, with $\mathrm{dyg}(\beta) = 6$. Since its Parry polynomial is irreducible of degree 24 it is equal to the minimal polynomial of β: $P_{\beta,P} = P_\beta = P_{\beta,P}^*$. Since β is not a simple Parry number, and has preperiod length 1, Theorem 4.14 implies that $U_\beta(z) = -(1 - z^{23})$; in this case the radius of convergence R_β of $U_\beta(z)$ is infinite. Any other Salem number β of Table 1 has beta-conjugates γ ; then, for these Salem numbers, by Theorem 4.14, $U_\beta(z)$ is meromorphic in $\mathbb{C}$ and holomorphic in the open disk $D(0, R_\beta)$ with $R_\beta = \min\big\{1, \min\big\{|\gamma|^{-1} \mid \gamma \text{ beta-conjugate of } \beta\big\}\big\}$.

Definition 5.7. Let $\beta > 1$ be an algebraic integer such that $\mathrm{M}(\beta) < \Theta$. The minimal polynomial $P_\beta(X)$ is said to be fracturable if the power series $U_\beta(z)$ in (5.10) is not reduced to a constant.

The fracturability of the minimal polynomial P_β of $\beta > 1$ close to one is very often the rule, at small Mahler measure. In the theory of divisibility of polynomials over a field, in Commutative Algebra, the minimal polynomials are the irreducible elements on which the (classical) theory of ideals is constructed. These irreducible elements are now canonically fracturable. This type of fracturability of the minimal polynomials implies a new theory of ideals in extensions of rings of polynomials when $\beta > 1$ is close to one. The author will develop it further later.

5.3 A lenticulus of zeroes of $f_\beta(z)$ in the cusp of Solomyak's fractal

In this subsection $\beta \in (1, \theta_6^{-1})$ is assumed to be a real number (algebraic or transcendental) such that $\beta \notin \{\theta_n^{-1} \mid n \geq 6\}$. In Theorem 5.15 it will be proved that, to such

a β, is associated a lenticulus of zeroes of $f_\beta(z)$ in the cusp of Solomyak's fractal $\mathscr{G}$, complementing Theorem 4.19.

Lenticuli of zeroes of $f_\beta(z)$ were already proved to exist when $\beta \in \{\theta_n^{-1} \mid n \geq 12\}$. They were used to give a direct proof of the Conjecture of Lehmer for $\{\theta_n^{-1} \mid n \geq 2\}$ in [VG6]. In the sequel we consider the remaining cases of βs, including all algebraic integers $\beta > 1$.

The method which will be used to detect the lenticuli of zeroes of $f_\beta(z)$ is the method of Rouché. This method will be shown to be powerful enough to reach relevant minorants of the Mahler measure $\mathrm{M}(\beta)$ for $\beta > 1$ any algebraic integer (§ 5.4).

Let $n := \mathrm{dyg}(\beta)$. The algebraic integers $z_{j,n}, 1 \leq j < \lfloor n/6 \rfloor$, which constitute the lenticulus $\mathscr{L}_{\theta_n^{-1}}$ in the upper Poincaré half-plane satisfy (§4.4):

$$f_{\theta_n^{-1}}(\theta_n) = f_{\theta_n^{-1}}(z_{1,n}) = f_{\theta_n^{-1}}(z_{2,n}) = f_{\theta_n^{-1}}(z_{3,n}) = \ldots = f_{\theta_n^{-1}}(z_{\lfloor n/6 \rfloor,n}) = 0,$$

with $f_{\theta_n^{-1}}(z) = -1 + z + z^n$. The Parry Upper function at β is characterized by the sequence of exponents $(m_q)_{q \geq 0}$:

$$f_\beta(z) = -1 + z + z^n + z^{m_1} + z^{m_2} + z^{m_3} + \ldots = G_n(z) + \sum_{q \geq 1} z^{m_q}, \tag{5.14}$$

where $m_0 := n$, with the fundamental minimal gappiness condition:

$$m_{q+1} - m_q \geq n - 1 \qquad \text{for all } q \geq 0. \tag{5.15}$$

The Rényi β-expansion $d_\beta(1)$ of 1 is infinite or not, namely the sequence of exponents $(m_q)_{q \geq 0}$ is either infinite or finite: if it is infinite the integers m_q never take the value $+\infty$; if not the power series $f_\beta(z)$ is a polynomial of degree m_q for some integer $m_q, q \geq 2$. In both cases, the integer $m_1 \geq 2n - 1$ is finite.

We will compute real numbers $t_{j,n} \in (0,1)$ such that the small circles $C_{j,n} := \{z \mid |z - z_{j,n}| = \frac{t_{j,n}}{n}\}$ of respective centers $z_{j,n}$, $|z_{j,n}| < 1$, all satisfy the Rouché conditions:

$$\left|f_\beta(z) - G_n(z)\right| = \left|\sum_{q \geq 1} z^{m_q}\right| < |G_n(z)| \qquad \text{for } z \in C_{j,n}, \text{ for } j = 1, 2, \ldots, J_n, \tag{5.16}$$

are pairwise disjoint, are small enough to avoid to intersect $|z| = 1$, with $J_n \leq \lfloor \frac{n}{6} \rfloor$ the largest possible integer (in the sense of Definition 5.11 and Proposition 5.12). As a consequence, the number of zeroes of $f_\beta(z)$ and $G_n(z)$ in the open disc $D_{j,n} := \{z \mid |z - z_{j,n}| < \frac{t_{j,n}}{n}\}$ will be equal, implying the existence of a simple zero of the Parry Upper function $f_\beta(z)$ in each disc $D_{j,n}$. The maximality of J_n means that the conditions of Rouché cannot be satisfied as soon as $J_n < j \leq \lfloor \frac{n}{6} \rfloor$ for the reason that the circles $C_{j,n}$ are too close to $|z| = 1$.

The values $t_{j,n}$ are necessarily smaller than π in order to avoid any overlap between two successive circles $C_{j,n}$ and $C_{j+1,n}$. Indeed, since the argument $\arg z_{j,n}$ of the j-th root $z_{j,n}$ is roughly equal to $2\pi j/n$ (Proposition 3.7), the distance $|z_{j,n} - z_{j+1,n}|$ is approximately $2\pi/n$.

The problem of the choice of the radius $t_{j,n}/n$ is a true problem. On one hand, a too small radius would lead to make impossible the application of the Rouché conditions, in particular for those discs $C_{j,n}$ located very near the unit circle. Indeed, we do not know a priori whether the unit circle is a natural boundary or not for $f_\beta(z)$; locating zeroes close to a natural boundary is a difficult problem in general. On the other hand, taking larger values of $t_{j,n}$ readily leads to a bad localization of the zeroes of $f_\beta(z)$, and hence, for algebraic integers $\beta > 1$, to a trivial minoration of the Mahler measure $\mathrm{M}(\beta)$. The sequel reports a compromise, after many trials of the author, which works (§ 5.4).

For any real number $\beta \in (1, \theta_6^{-1})$ such that $\beta \not\in \{\theta_n^{-1} \mid n \geq 6\}$ let us denote by $\omega_{j,n} \in D_{j,n}$ the simple zero of $f_\beta(z)$; then $|\omega_{j,n}| < 1$ and

$$|\omega_{j,n}| \;\leq\; |z_{j,n}| + \frac{t_{j,n}}{n} \qquad \text{with} \quad z_{j,n} \;\neq\; \omega_{j,n}, \quad j = 1, 2, \ldots, J_n; \tag{5.17}$$

if, in addition, $\beta > 1$ is an algebraic integer, the strategy for obtaining a minorant of $\mathrm{M}(\beta)$ will be the following: to identify the zeroes $\omega_{j,n}$ as roots of the minimal polynomial $P^*_\beta(z) = P_\beta(z)$, then to obtain a lower bound of the Mahler measure $\mathrm{M}(\beta)$ (will be made explicit in §5.4) from these roots by

$$\beta \times \prod |\omega_{j,n}|^{-2} \;\;\geq\;\; \theta_n^{-1} \times \prod_j (|z_{j,n}| + \frac{t_{j,n}}{n})^{-2}, \tag{5.18}$$

where j runs over $\{1, 2, \ldots, J_n\}$.

In general, for any real number $\beta \in (1, \theta_6^{-1})$ such that $\beta \not\in \{\theta_n^{-1} \mid n \geq 6\}$, the quantities $t_{j,n}$ will be estimated by the following inequalities:

$$\frac{|z|^{2(n-1)+1}}{1 - |z|^{n-1}} = \frac{|z|^{2n-1}}{1 - |z|^{n-1}} \;<\; |G_n(z)| \quad \text{for } z \in C_{j,n}, \quad j = 1, 2, \ldots, J_n \tag{5.19}$$

instead of (5.16), too complicated to handle. In (5.19) the exponent "$n-1$" comes from the minimal gappiness condition (5.15), that is from the dynamical degree n of β itself, as unique variable. Indeed, due to the great variety of possible infinite admissible sequences $(m_q)_{q\geq 1}$ in the power series $f_\beta(z)$ in (5.14), for which $m_q \geq q(n-1) + n$ for all $q \geq 1$, we will proceed by taking the upper bound condition (5.19) which comes from the general inequality:

$$\bigl|f_\beta(z) - G_n(z)\bigr| = \left|\sum_{q\geq 1} z^{m_q}\right| \leq \sum_{q\geq 1} |z^{m_q}| \leq \frac{|z|^{2n-1}}{1 - |z|^{n-1}}, \qquad |z| < 1.$$

The radius $t_{0,n}/n$ of the first circle $C_{0,n} := \{z \mid |z - \theta_n| = \frac{t_{0,n}}{n}\}$, which contains β^{-1}, is readily obtained without the method of Rouché.

Lemma 5.8. *Let $n \geq 7$.*

$$t_{0,n} := \left(\frac{\operatorname{Log}\operatorname{Log} n}{\operatorname{Log} n}\right)^2.$$

Proof. Since β^{-1} runs over the open interval (θ_{n-1}, θ_n), this interval (θ_{n-1}, θ_n) is necessarily completely included in $D_{0,n}$, and the radius of $C_{0,n}$ is $\theta_n - \theta_{n-1}$. We deduce the result from Lemma 5.1. From Proposition 3.6 the root $z_{1,n}$ admits $\Im(z_{1,n}) = \frac{2\pi}{n}(1 - \frac{1}{\mathrm{Log}\, n} + \ldots)$ as imaginary part. Then, for any $t_{1,n} \in (0,1)$, the circle $C_{0,n}$, of radius $t_{0,n}/n$, and $C_{1,n}$ are disjoint and do not intersect $|z| = 1$. □

By Proposition 3.3 the only angular sector to be considered for the roots $z_{j,n}$ of G_n and the Rouché circles $C_{j,n}$, up to complex-conjugation, is $0 \leq \arg(z) \leq +\frac{\pi}{3}$. In this sector the "bump" angular sector, $\arg z \in (0, 2\pi(\mathrm{Log}\, n)/n)$ (cf Appendix; Remark 3.3 in [VG6]), will be shown to contribute negligibly.

The existence of the roots $\omega_{j,n}$ in the main subsector is proved in Theorem 5.9, then in Theorem 5.15 in a refined version. Proposition 5.14 completes the proof of their existence in the bump angular sector. In the complement of the family of the adjustable Rouché discs Theorem 5.21 asserts the existence of a zerofree region depending upon the dynamical degree of β.

In the following we consider the problem of the parametrization of the radii $t_{j,n}/n$ by a unique real number $a \geq 1$, allowing to adjust continuously and uniformly the size of each circle $C_{j,n}$. We solve it by finding an optimal value.

Theorem 5.9. *Let $n \geq n_1 = 195$, $a \geq 1$, and $j \in \{\lceil v_n \rceil, \lceil v_n \rceil + 1, \ldots, \lfloor n/6 \rfloor\}$. Denote by $C_{j,n} := \{z \mid |z - z_{j,n}| = \frac{t_{j,n}}{n}\}$ the circle centered at the j-th root $z_{j,n}$ of $-1 + X + X^n$, with $t_{j,n} = \frac{\pi |z_{j,n}|}{a}$. Then the condition of Rouché*

$$\frac{|z|^{2n-1}}{1 - |z|^{n-1}} < |-1 + z + z^n|, \qquad \text{for all } z \in C_{j,n}, \tag{5.20}$$

holds true on the circle $C_{j,n}$ for which the center $z_{j,n}$ satisfies

$$\frac{|-1 + z_{j,n}|}{|z_{j,n}|} < \frac{1 - \exp\left(\frac{-\pi}{a}\right)}{2\exp\left(\frac{\pi}{a}\right) - 1}. \tag{5.21}$$

The condition $n \geq 195$ ensures the existence of such roots $z_{j,n}$. Taking the value $a = a_{\max} = 5.87433\ldots$ for which the upper bound of (5.21) *is maximal, equal to* $0.171573\ldots$, *the roots $z_{j,n}$ which satisfy* (5.21) *all belong to the angular sector, independent of n:*

$$\arg(z) \in \left[0, +\frac{\pi}{18.2880}\right]. \tag{5.22}$$

For any real number $\beta > 1$ having $\mathrm{dyg}(\beta) = n$, *$f_\beta(z)$ admits a simple zero $\omega_{j,n}$ in $D_{j,n}$ for which the center $z_{j,n}$ satisfies* (5.21) *with $a = a_{\max}$, and j in the range* $\{\lceil v_n \rceil, \lceil v_n \rceil + 1, \ldots, \lfloor n/6 \rfloor\}$.

Proof. Denote by $\varphi := \arg(z_{j,n})$ the argument of the j-th root $z_{j,n}$. Since $-1 + z_{j,n} + z_{j,n}^n = 0$, we have $|z_{j,n}|^n = |-1 + z_{j,n}|$. Let us write $z = z_{j,n} + \frac{t_{j,n}}{n} e^{i\psi} = z_{j,n}(1 + \frac{\pi}{an} e^{i(\psi - \varphi)})$ the generic element belonging to $C_{j,n}$, with $\psi \in [0, 2\pi]$. Let $X := \cos(\psi -$

φ). Let us show that if the inequality (5.20) of Rouché holds true for $X = +1$, for a certain circle $C_{j,n}$, then it holds true for all $X \in [-1,+1]$, that is for every argument $\psi \in [0,2\pi]$, i.e. for every $z \in C_{j,n}$. Let us show

$$\left|1+\frac{\pi}{an}e^{i(\psi-\varphi)}\right|^n = \exp\Big(\frac{\pi X}{a}\Big) \times \left(1-\frac{\pi^2}{2a^2 n}(2X^2-1)+O(\frac{1}{n^2})\right)$$

and

$$\arg\left(\left(1+\frac{\pi}{an}e^{i(\psi-\varphi)}\right)^n\right) = sgn(\sin(\psi-\varphi)) \times \left(\frac{\pi\sqrt{1-X^2}}{a}[1-\frac{\pi X}{an}]+O(\frac{1}{n^2})\right).$$

Indeed, since $\sin(\psi-\varphi) = \pm\sqrt{1-X^2}$, then

$$\left(1+\frac{\pi}{an}e^{i(\psi-\varphi)}\right)^n = \exp\left(n\,\mathrm{Log}\,(1+\frac{\pi}{an}e^{i(\psi-\varphi))})\right)$$

$$= \exp\left(\frac{\pi}{a}(X\pm i\sqrt{1-X^2})+\left[-\frac{n}{2}(\frac{\pi}{an}(X\pm i\sqrt{1-X^2}))^2+O(\frac{1}{n^2})\right]\right)$$

$$= \exp\Big(\frac{\pi X}{a}-\frac{\pi^2}{2a^2 n}(2X^2-1)+O(\frac{1}{n^2})\Big) \times \exp\left(\pm i\,\frac{\pi\sqrt{1-X^2}}{a}[1-\frac{\pi X}{an}]+O(\frac{1}{n^2})\right).$$

Moreover,

$$\left|1+\frac{\pi}{an}e^{i(\psi-\varphi)}\right| = \left|1+\frac{\pi}{an}(X\pm i\sqrt{1-X^2})\right| = 1+\frac{\pi X}{an}+O(\frac{1}{n^2}).$$

with

$$\arg(1+\frac{\pi}{an}e^{i(\psi-\varphi)}) = sgn(\sin(\psi-\varphi)) \times \frac{\pi\sqrt{1-X^2}}{an}+O(\frac{1}{n^2}).$$

For all $n \geq 18$ (Proposition 3.5 in [VG6]), let us recall that

$$|z_{j,n}| = 1+\frac{1}{n}\mathrm{Log}\,(2\sin\frac{\pi j}{n})+\frac{1}{n}O\left(\frac{\mathrm{Log}\,\mathrm{Log}\,n}{\mathrm{Log}\,n}\right)^2. \tag{5.23}$$

Then the left-hand side term of (5.20) is

$$\frac{|z|^{2n-1}}{1-|z|^{n-1}} = \frac{|-1+z_{j,n}|^2\left|1+\frac{\pi}{an}e^{i(\psi-\varphi)}\right|^{2n}}{|z_{j,n}|\,\left|1+\frac{\pi}{an}e^{i(\psi-\varphi)}\right|-|-1+z_{j,n}|\,\left|1+\frac{\pi}{an}e^{i(\psi-\varphi)}\right|^n}$$

$$= \frac{|-1+z_{j,n}|^2\left(1-\frac{\pi^2}{an}(2X^2-1)\right)\exp(\frac{2\pi X}{a})}{\left(1+\frac{1}{n}\mathrm{Log}\,(2\sin\frac{\pi j}{n})+\frac{\pi X}{an}\right)-|-1+z_{j,n}|\left(1-\frac{\pi^2}{2an}(2X^2-1)\right)\exp(\frac{\pi X}{a})} \tag{5.24}$$

up to $\frac{1}{n} O\left(\frac{\mathrm{Log}\,\mathrm{Log}\, n}{\mathrm{Log}\, n}\right)^2$ -terms (in the terminant). The right-hand side term of (5.20) is

$$|-1+z+z^n| = \left|-1+z_{j,n}\Big(1+\frac{\pi}{na}e^{i(\psi-\varphi)}\Big)+z_{j,n}^n\Big(1+\frac{\pi}{na}e^{i(\psi-\varphi)}\Big)^n\right|$$

$$= \Bigg| -1+z_{j,n}(1 \pm i\frac{\pi\sqrt{1-X^2}}{an})(1+\frac{\pi X}{an})+$$

$$(1-z_{j,n})\big(1-\frac{\pi^2}{2a^2 n}(2X^2-1)\big)\exp(\frac{\pi X}{a})\,\exp\Big(\pm i\Big(\frac{\pi\sqrt{1-X^2}}{a}[1-\frac{\pi X}{an}]\Big)\Big)+O(\frac{1}{n^2})\Bigg| \tag{5.25}$$

Let us consider (5.24) and (5.25) at the first order for the asymptotic expansions, i.e. up to $O(1/n)$ - terms instead of up to $O(\frac{1}{n}(\mathrm{Log}\,\mathrm{Log}\, n/\mathrm{Log}\, n)^2)$ - terms or $O(1/n^2)$ - terms. (5.24) becomes:

$$\frac{|-1+z_{j,n}|^2\exp(\frac{2\pi X}{a})}{|z_{j,n}|-|-1+z_{j,n}|\exp(\frac{\pi X}{a})}$$

and (5.25) is equal to:

$$|-1+z_{j,n}|\left|1-\exp(\frac{\pi X}{a})\,\exp\Big(\pm i\,\frac{\pi\sqrt{1-X^2}}{a}\Big)\right|$$

and is independent of the sign of $\sin(\psi-\varphi)$. Then the inequality (5.20) is equivalent to

$$\frac{|-1+z_{j,n}|^2\exp(\frac{2\pi X}{a})}{|z_{j,n}|-|-1+z_{j,n}|\exp(\frac{\pi X}{a})} < |-1+z_{j,n}|\left|1-\exp(\frac{\pi X}{a})\,\exp\Big(\pm i\,\frac{\pi\sqrt{1-X^2}}{a}\Big)\right|, \tag{5.26}$$

and (5.26) to

$$\frac{|-1+z_{j,n}|}{|z_{j,n}|} < \frac{\left|1-\exp(\frac{\pi X}{a})\,\exp\Big(i\,\frac{\pi\sqrt{1-X^2}}{a}\Big)\right|\exp(\frac{-\pi X}{a})}{\exp(\frac{\pi X}{a})+\left|1-\exp(\frac{\pi X}{a})\,\exp\Big(i\,\frac{\pi\sqrt{1-X^2}}{a}\Big)\right|} =: \kappa(X,a). \tag{5.27}$$

Denote by $\kappa(X,a)$ the right-hand side term, as a function of (X,a), on $[-1,+1]\times[1,+\infty)$. It is routine to show that, for any fixed a, the partial derivative $\partial\kappa_X$ of $\kappa(X,a)$ with respect to X is strictly negative on the interior of the domain. The function $x\to\kappa(x,a)$ takes its minimum at $X=1$, and this minimum is always strictly positive. Hence the inequality of Rouché (5.20) will be satisfied on $C_{j,n}$ once it is satisfied at $X=1$.

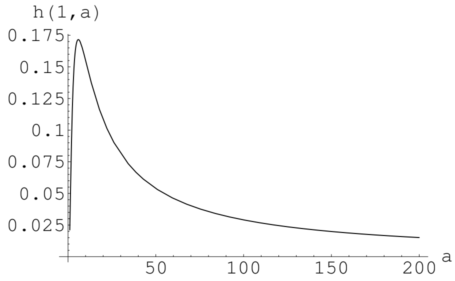

Figure 2. Curve of the Rouché condition $a \to \kappa(1,a)$ (upper bound in (5.21)), for the circles $C_{j,n} = \{z \mid |z - z_{j,n}| = \pi |z_{j,n}|/(an)\}$ centered at the zeroes $z_{j,n}$ of the trinomial $-1+X+X^n$, as a function of the size of the circles $C_{j,n}$ parametrized by the adjustable real number $a \geq 1$.

For which range of values of j/n? Up to $O(1/n)$-terms in (5.27), it is given by the set of integers j for which $z_{j,n}$ satisfies:

$$\frac{|-1+z_{j,n}|}{|z_{j,n}|} < \kappa(1,a) = \frac{\left|1-\exp\left(\frac{\pi}{a}\right)\right|\exp\left(\frac{-\pi}{a}\right)}{\exp\left(\frac{\pi}{a}\right)+\left|1-\exp\left(\frac{\pi}{a}\right)\right|}. \tag{5.28}$$

In order to take into account a collection of roots of $z_{j,n}$ as large as possible, i.e. in order to have a minorant of the Mahler measure $\mathrm{M}(\beta)$ the largest possible, the value of $a \geq 1$ has to be chosen such that $a \to \kappa(1,a)$ is maximal in (5.28).

The function $a \to \kappa(1,a)$ reaches its maximum $\kappa(1,a_{\max}) := 0.171573\ldots$ at $a_{\max} = 5.8743\ldots$ (Figure 2). Denote by J_n the maximal integer j for which (5.28) is satisfied and in which a is taken equal to $a_{\max}$ (Definition 5.11 and Proposition 5.12). From Proposition 5.12, in which are reported the asymptotic expansions of J_n and $\arg(z_{J_n,n})$, we deduce

$$\arg(z_{j,n}) < \frac{\pi}{18.2880\ldots} = 0.171784\ldots \quad \text{for } j = \lceil v_n \rceil, \lceil v_n \rceil + 1, \ldots, J_n. \tag{5.29}$$

Remark 5.10. The minimal value $n_1 = 195$ is calculated by the condition $2\pi \frac{v_n}{n} < \frac{\pi}{18.2880...} = 0.171784\ldots$, for all $n \geq n_1$, for having a strict inclusion, of the "bump sector" inside the angular sector defined by the maximal opening angle $0.171784\ldots$ (cf Appendix for the sequence (v_n))

This finishes the proof. □

Let us calculate the argument of the last root $z_{j,n}$ for which (5.27) is an equality with $X = 1$.

Definition 5.11. Let $n \geq 195$. Denote by J_n the largest integer $j \geq 1$ such that the root $z_{j,n}$ of G_n satisfies

$$\frac{|-1+z_{j,n}|}{|z_{j,n}|} \;\leq\; \kappa(1,a_{\max}) = \frac{1-\exp\big(\frac{-\pi}{a_{\max}}\big)}{2\exp\big(\frac{\pi}{a_{\max}}\big)-1} = 0.171573\ldots \tag{5.30}$$

Let us observe that the upper bound $\kappa(1,a_{\max})$ is independent of n. From this independence we deduce the following “limit” angular sector in which the Rouché conditions can be applied.

Proposition 5.12. *Let $n \geq 195$. Let us put $\kappa := \kappa(1,a_{\max})$ for short. Then*

$$\arg(z_{J_n,n}) = 2\arcsin\big(\frac{\kappa}{2}\big) + \frac{\kappa \operatorname{Log}\kappa}{n\sqrt{4-\kappa^2}} + \frac{1}{n}O\big(\big(\frac{\operatorname{Log}\operatorname{Log} n}{\operatorname{Log} n}\big)^2\big), \tag{5.31}$$

$$J_n = \frac{n}{\pi}\big(\arcsin\big(\frac{\kappa}{2}\big)\big) + \frac{\kappa \operatorname{Log}\kappa}{\pi\sqrt{4-\kappa^2}} + O\big(\big(\frac{\operatorname{Log}\operatorname{Log} n}{\operatorname{Log} n}\big)^2\big) \tag{5.32}$$

with, at the limit,

$$\lim_{n\to+\infty} \arg(z_{J_n,n}) = \lim_{n\to+\infty} 2\pi\frac{J_n}{n} = 2\arcsin\big(\frac{\kappa}{2}\big) = 0.171784\ldots \tag{5.33}$$

Proof. Since $\lim_{n\to+\infty}|z_{J_n,n}| = 1$, we deduce from (5.30) that the limit argument φ_{lim} of $z_{J_n,n}$ satisfies $|-1+\cos(\varphi_{lim})+i\sin(\varphi_{lim})| = 2\sin(\varphi_{lim}/2) = \kappa(1,a_{\max})$. We deduce (5.33), and the equality between the two limits from (5.34).

From (5.30), the inequality $|-1+z_{j,n}| \leq |z_{j,n}|\,\kappa(1,a_{\max})$ already implies that $\arg(z_{J_n,n})) < \varphi_{lim}$. In the sequel, we will use the asymptotic expansions of the roots $z_{J_n,n}$. From Section 6 in [VG6] the argument of $z_{J_n,n}$ takes the following form

$$\arg(z_{J_n,n})) = 2\pi\big(\frac{J_n}{n} + \Re\big) \quad \text{with} \quad \Re = -\frac{1}{2\pi n}\left[\frac{1-\cos(\frac{2\pi J_n}{n})}{\sin(\frac{2\pi J_n}{n})}\operatorname{Log}\big(2\sin\big(\frac{\pi J_n}{n}\big)\big)\right] \tag{5.34}$$

with

$$\operatorname{tl}(\arg(z_{J_n,n}))) = +\frac{1}{n}O\left(\left(\frac{\operatorname{Log}\operatorname{Log} n}{\operatorname{Log} n}\right)^2\right).$$

Its modulus is

$$|z_{J_n,n}| = 1 + \frac{1}{n}\operatorname{Log}\big(2\sin\frac{\pi J_n}{n}\big) + \frac{1}{n}O\left(\frac{\operatorname{Log}\operatorname{Log} n}{\operatorname{Log} n}\right)^2. \tag{5.35}$$

Denote $\varphi := \arg(z_{J_n,n})$. Up to $\frac{1}{n}O\big(\big(\frac{\operatorname{Log}\operatorname{Log} n}{\operatorname{Log} n}\big)^2\big)$-terms, we have

$$|-1+z_{J_n,n}|^2 = \left|-1+\big[1+\frac{1}{n}\operatorname{Log}\big(2\sin\frac{\pi J_n}{n}\big)\big](\cos(\varphi)+i\sin(\varphi))\right|^2$$

$$= [-1+[1+\frac{1}{n}\mathrm{Log}\,(2\sin\frac{\pi J_n}{n})](\cos(\varphi)]^2 + [1+\frac{1}{n}\mathrm{Log}\,(2\sin\frac{\pi J_n}{n})]^2(\sin(\varphi)^2$$

$$= 1+[1+\frac{1}{n}\mathrm{Log}\,(2\sin\frac{\pi J_n}{n})]^2 - 2[1+\frac{1}{n}\mathrm{Log}\,(2\sin\frac{\pi J_n}{n})]\cos(\varphi)$$

$$= 4(\sin(\frac{\varphi}{2}))^2 + \frac{4}{n}(\sin(\frac{\varphi}{2}))^2\,\mathrm{Log}\,(2\sin\frac{\pi J_n}{n}) = 4(\sin(\frac{\varphi}{2}))^2[1+\frac{1}{n}\mathrm{Log}\,(2\sin\frac{\pi J_n}{n})]. \tag{5.36}$$

Up to $\frac{1}{n}O\big((\frac{\mathrm{Log}\,\mathrm{Log}\,n}{\mathrm{Log}\,n})^2\big)$-terms, due to the definition of J_n, let us consider (5.30) as an equality; hence, from (5.36) and (5.35), the following identity should be satisfied

$$2\sin(\frac{\varphi}{2}) = \kappa\,[1+\frac{1}{2n}\mathrm{Log}\,(2\sin\frac{\pi J_n}{n})] \tag{5.37}$$

We now use (5.37) to obtain an asymptotic expansion of $\psi_n := 2\pi\frac{J_n}{n} - \varphi_{lim}$ as a function of n and φ_{lim} up to $\frac{1}{n}O\big((\frac{\mathrm{Log}\,\mathrm{Log}\,n}{\mathrm{Log}\,n})^2\big)$-terms. First, at the first order in ψ_n,

$$\sin(\frac{\pi J_n}{n}) = \frac{\psi_n}{2}\cos(\frac{\varphi_{lim}}{2}) + \sin(\frac{\varphi_{lim}}{2}), \quad \cos(\frac{\pi J_n}{n}) = -\frac{\psi_n}{2}\sin(\frac{\varphi_{lim}}{2}) + \cos(\frac{\varphi_{lim}}{2}),$$

$$\mathrm{Log}\,\big(2\sin(\frac{\pi J_n}{n})\big) = \mathrm{Log}\,\big(2\sin(\frac{\varphi_{lim}}{2})\big) + \psi_n\frac{\cos(\frac{\varphi_{lim}}{2})}{2\sin(\frac{\varphi_{lim}}{2})} = \mathrm{Log}\,\kappa + \psi_n\frac{\cos(\frac{\varphi_{lim}}{2})}{h}.$$

Moreover,

$$\Big[\frac{1-\cos(\frac{2\pi J_n}{n})}{\sin(\frac{2\pi J_n}{n})}\mathrm{Log}\,(2\sin(\frac{\pi J_n}{n}))\Big]$$

$$= \tan(\frac{\varphi_{lim}}{2})(\mathrm{Log}\,\kappa)\Big[1+\psi_n\Big(\frac{1}{\sin(\varphi_{lim})} + \frac{\cos(\frac{\varphi_{lim}}{2})}{\kappa\,\mathrm{Log}\,\kappa}\Big)\Big]. \tag{5.38}$$

Hence, with $2\sin(\varphi/2) = 2\sin(\pi J_n/n)\cos(\pi\Re) + 2\cos(\pi J_n/n)\sin(\pi\Re)$, and from (5.34), up to $\frac{1}{n}O\big((\frac{\mathrm{Log}\,\mathrm{Log}\,n}{\mathrm{Log}\,n})^2\big)$-terms, the identity (5.37) becomes

$$[\psi_n\cos(\frac{\varphi_{lim}}{2}) + 2\sin(\frac{\varphi_{lim}}{2})] + (\frac{-2\cos(\frac{\varphi_{lim}}{2})\tan(\frac{\varphi_{lim}}{2})\,\mathrm{Log}\,\kappa}{2n}) = \kappa\,\big[1+\frac{\mathrm{Log}\,\kappa}{2n}\big].$$

We deduce

$$\psi_n = \frac{\kappa\,\mathrm{Log}\,\kappa}{n\cos(\frac{\varphi_{lim}}{2})} + \frac{1}{n}O\big((\frac{\mathrm{Log}\,\mathrm{Log}\,n}{\mathrm{Log}\,n})^2\big), \tag{5.39}$$

then $2\pi J_n/n = \psi_n + \varphi_{lim}$ and (5.32). With $2\pi J_n/n$, and from (5.34) and (5.38) we deduce (5.31). This finishes the proof. $\square$

Remark 5.13. (i) The maximal half-opening angle of the sector in which one can detect zeroes of $f_\beta(z)$, for any β such that $\theta_{n-1} < \beta^{-1} < \theta_n$, by the method of Rouché,

is $0.17178... = 2\arcsin(\frac{\kappa(1,a_{\max})}{2})$. Remarkably this upper bound $2\arcsin(\frac{\kappa(1,a_{\max})}{2})$ is independent of n. By comparison it is fairly small with respect to $\pi/3$ for the Perron numbers θ_n^{-1}.

(ii) The curve $a \to \kappa(1,a)$, given by Figure 2, is such that any value in the interval $(0, \kappa(1, a_{max}))$ is reached by the function $\kappa(1,a)$ from two values say a_1 and a_2, of a, satisfying $a_1 < a_{max} < a_2$. On the contrary, the correspondence $a_{max} \leftrightarrow \kappa(1, a_{max})$ is unique, corresponding to a double root. Denote $D := \exp(\pi/a_{max})$ and $\kappa := \kappa(1, a_{max})$. It means that the quadratic algebraic equation $2\kappa D^2 - (\kappa+1)D + 1 = 0$ deduced from the upper bound in (5.30) has necessarily a discriminant equal to zero. The discriminant is $\kappa^2 - 6\kappa + 1$. Therefore $D = (\kappa+1)/(4\kappa)$ and the limit value $x = 2\arcsin(\kappa/2)$ in (5.33) satisfies the quadratic algebraic equation

$$4(\sin(x/2))^2 - 12\sin(x/2) + 1 = 0.$$

Proposition 5.14. *Let $n \geq n_1 = 195$. The circles $C_{j,n} := \{z \mid |z - z_{j,n}| = \frac{\pi |z_{j,n}|}{n\, a_{\max}}\}$ centered at the roots $z_{j,n}$ of the trinomial $-1+z+z^n$ which belong to the "bump sector", namely for $j \in \{1,2,\ldots,\lfloor v_n \rfloor\}$, are such that the conditions of Rouché*

$$\frac{|z|^{2n-1}}{1-|z|^{n-1}} < |-1+z+z^n|, \quad \text{for all } z \in C_{j,n}, \quad 1 \leq j \leq \lfloor v_n \rfloor, \tag{5.40}$$

hold true. For any real number $\beta > 1$ having $\mathrm{dyg}(\beta) = n$*, $f_\beta(z)$ admits a simple zero $\omega_{j,n}$ in $D_{j,n}$ (with $a = a_{\max}$), for j in the range $\{1,2,\ldots,\lfloor v_n \rfloor\}$.*

Proof. The development terms "D" of the asymptotic expansions of $|z_{j,n}|$ change from the main angular sector $\arg z \in (2\pi(\mathrm{Log}\, n)/n, \pi/3)$ to the first transition region $\arg z \asymp 2\pi(\mathrm{Log}\, n)/n$, the "bump sector", further to the second transition region $\arg z \asymp 2\pi\sqrt{(\mathrm{Log}\, n)(\mathrm{Log}\,\mathrm{Log}\, n)}/n$, and to a small neighbourhood of θ_n (Section 3.2).

Then the proof of (5.40) is the same as that of Theorem 5.9 once (5.23) is substituted by the suitable asymptotic expansions which correspond to the angular sector of the "bump". The terminants of the respective asymptotic expansions of $|z_{j,n}|$ also change: this change imposes to reconsider (5.24) and (5.25) up to $\mathrm{Log}\, n/n$ - terms, and not up to $1/n$ - terms, as in the proof of Theorem 5.9. It is remarkable that the inequality (5.27) remains the same, with the same upper bound function $\kappa(X,a)$. Then the equation of the curve of the Rouché condition $a \to \kappa(1,a)$, on $[1,+\infty)$, is the same as in Theorem 5.9 for controlling the conditions of Rouché. The optimal value $a_{\max}$ of a also remains the same, and (5.20) also holds true for those $z_{j,n}$ in the bump sector. ☐

From the inequalities (5.21) in Theorem 5.9, also used in the proof of Proposition 5.14, we now obtain a finer localization of a subcollection of the roots $\omega_{j,n}$ of the Parry Upper function $f_\beta(z)$, and a definition of the lenticulus $\mathscr{L}_\beta$ of β, as follows.

Theorem 5.15. *Let $n \geq n_1 = 195$. Let $\beta > 1$ be any real number having* $\mathrm{dyg}(\beta) = n$*. The Parry Upper function $f_\beta(z)$ has an unique simple zero $\omega_{j,n}$ in each disc $D_{j,n} :=$*

$\{z \mid |z - z_{j,n}| < \frac{\pi |z_{j,n}|}{n\, a_{\max}}\}$, $j = 1, 2, \ldots, J_n$, *which satisfies the additional inequality:*

$$|\omega_{j,n} - z_{j,n}| < \frac{\pi |z_{j,n}|}{n\, a_{j,n}} \qquad \textit{for} \;\; j = \lceil v_n \rceil, \lceil v_n \rceil + 1, \ldots, J_n, \tag{5.41}$$

where $a_{J_n,n} = a_{\max}$ *and, for* $j = \lceil v_n \rceil, \ldots, J_n - 1$, *the value* $a_{j,n}$ *,* $> a_{\max}$*, is defined by*

$$D\Big(\frac{\pi}{a_{j,n}}\Big) \;=\; \mathrm{Log}\Big[\frac{1 + B_{j,n} - \sqrt{1 - 6B_{j,n} + B_{j,n}^2}}{4B_{j,n}}\Big] \tag{5.42}$$

$$\textit{with} \qquad B_{j,n} := 2\sin\Big(\frac{\pi j}{n}\Big)\Big(1 - \frac{1}{n}\mathrm{Log}\,(2\sin(\frac{\pi j}{n}))\Big),$$

and, putting $D := D\big(\frac{\pi}{a_{j,n}}\big)$ *for short,*

$$\mathrm{tl}\Big(\frac{\pi}{a_{j,n}}\Big) \;=\; \frac{2}{n} \times B_{j,n}^{-1}\Big(\frac{-3 + \exp(-D) + 2\exp(D)}{4 - \exp(-D) - 2\exp(D)}\Big) \times \Big(\frac{\mathrm{Log}\,\mathrm{Log}\, n}{\mathrm{Log}\, n}\Big)^2. \tag{5.43}$$

An upper bound of the tails, independent of j, is given by

$$O\Big(\frac{(\mathrm{Log}\,\mathrm{Log}\, n)^2}{(\mathrm{Log}\, n)^3}\Big) \tag{5.44}$$

with the constant $\frac{1}{7\pi}$ *in the Big O. The lenticulus* $\mathscr{L}_\beta$ *associated with* β *is constituted by the following subset of roots of* $f_\beta(z)$*:*

$$\mathscr{L}_\beta := \{1/\beta\} \;\cup\; \bigcup_{j=1}^{J_n} (\{\omega_{j,n}\} \cup \{\overline{\omega_{j,n}}\}). \tag{5.45}$$

Proof. The existence of the zeroes comes from Proposition 5.14 and Theorem 5.9, with the maximal value J_n of the index j given by Proposition 5.11. To refine the localization of $\omega_{j,n}$ in the neighbourhood of $z_{j,n}$, in the main angular sector, i.e. for $j \in \{\lceil v_n \rceil, \lceil v_n \rceil + 1, \ldots, J_n\}$, the conditions of Rouché (5.20) are now used to define the new radii.

The value $a_{j,n}$ is defined by the development term $\mathrm{D}(\frac{\pi}{a_{j,n}})$, itself defined as follows:

$$\mathrm{D}\left(\frac{|-1 + z_{j,n}|}{|z_{j,n}|}\right) \;\;=:\;\; \frac{1 - \exp\big(-\mathrm{D}(\frac{\pi}{a_{j,n}})\big)}{2\exp\big(\mathrm{D}(\frac{\pi}{a_{j,n}})\big) - 1} \tag{5.46}$$

and the tail $\mathrm{tl}(\frac{\pi}{a_{j,n}})$ calculated from $\mathrm{tl}\left(\frac{|-1+z_{j,n}|}{|z_{j,n}|}\right)$ so that the Rouché condition

$$\frac{|-1 + z_{j,n}|}{|z_{j,n}|} = \mathrm{D}\left(\frac{|-1 + z_{j,n}|}{|z_{j,n}|}\right) + \mathrm{tl}\left(\frac{|-1 + z_{j,n}|}{|z_{j,n}|}\right) \;\;<\;\; \frac{1 - \exp\big(-\frac{\pi}{a_{j,n}}\big)}{2\exp\big(\frac{\pi}{a_{j,n}}\big) - 1} \tag{5.47}$$

holds true. From Proposition 3.11, denote

$$B_{j,n} := \mathrm{D}\left(\frac{|-1+z_{j,n}|}{|z_{j,n}|}\right) = 2\sin(\frac{\pi j}{n})\Big(1 - \frac{1}{n}\mathrm{Log}\,(2\sin(\frac{\pi j}{n}))\Big).$$

Let $W := \exp(\mathrm{D}(\frac{\pi}{a_{j,n}}))$. The identity (5.46) transforms into the equation of degree 2:

$$2B_{j,n}W^2 - \Big(B_{j,n}+1\Big)W + 1 \;=\; 0 \tag{5.48}$$

from which (5.49) is deduced. For the calculation of $\mathrm{tl}(\frac{\pi}{a_{j,n}})$, denote $D := \mathrm{D}(\frac{\pi}{a_{j,n}})$ and $tl_{j,n} := \mathrm{tl}(\frac{\pi}{a_{j,n}})$. Then, at the first order, $\frac{1-\exp\left(\frac{-\pi}{a_{j,n}}\right)}{2\exp\left(\frac{\pi}{a_{j,n}}\right)-1}$

$$= \frac{1-\exp\big(-D-tl_{j,n}\big)}{2\exp\big(D+tl_{j,n}\big)-1} = B_{j,n}[1+tl_{j,n}\times(\frac{4-\exp(-D)-2\exp(D)}{-3+\exp(-D)+2\exp(D)})].$$

From (5.47) and (3.17) the following inequality should be satisfied, with the constant 2 in the Big O,

$$\frac{1}{n}O\Big((\frac{\mathrm{Log}\,\mathrm{Log}\,n}{\mathrm{Log}\,n})^2\Big) = \mathrm{tl}\Big(\frac{|-1+z_{j,n}|}{|z_{j,n}|}\Big) \;<\; tl_{j,n}\times B_{j,n}(\frac{4-\exp(-D)-2\exp(D)}{-3+\exp(-D)+2\exp(D)})].$$

The expression of $tl_{j,n}$ in (5.43) follows, to obtain a strict inequality in (5.47). By (5.42) the quantity $\exp(D)$ is a function of $B_{j,n}$, which tends to $\frac{3}{4}$ when $B_{j,n}$ tends to 0; hence, at the first order, a lower bound of the function

$$B_{j,n} \to \Big|B_{j,n}(\frac{4-\exp(-D)-2\exp(D)}{-3+\exp(-D)+2\exp(D)})\Big|$$

is obtained for $j = \lceil v_n \rceil$, and given by $2\pi\frac{\mathrm{Log}\,n}{n}\times 7$. Then it suffices to take

$$tl_{j,n} = cste\Big(\frac{(\mathrm{Log}\,\mathrm{Log}\,n)^2}{(\mathrm{Log}\,n)^3}\Big)$$

with $cste = 1/(7\pi)$, to obtain a tail independent of j, and therefore the conditions of Rouché (5.47) satisfied with these new smaller radii and tails in the main angular sector. □

Remark 5.16. For n very large, up to second-order terms, (5.48) reduces to

$$4\sin(\frac{\pi j}{n})W^2 - \Big(2\sin(\frac{\pi j}{n})+1\Big)W + 1 \;=\; 0$$

and (5.42) to

$$D(\frac{\pi}{a_{j,n}}) \;=\; \mathrm{Log}\Big[\frac{1+2\sin(\frac{\pi j}{n}) - \sqrt{1-12\sin(\frac{\pi j}{n})+4(\sin(\frac{\pi j}{n}))^2}}{8\sin(\frac{\pi j}{n})}\Big]. \tag{5.49}$$

Lemma 5.17. *Let $n \geq 195$ and c_n defined by $|z_{J_n,n}| = 1 - \frac{c_n}{n}$. Let us put $\kappa := \kappa(1, a_{\max})$ for short. Then*

$$c_n = -(\operatorname{Log}\kappa)\,(1 + \frac{1}{n}) + \frac{1}{n} O\Big(\big(\frac{\operatorname{Log}\operatorname{Log} n}{\operatorname{Log} n}\big)^2\Big), \tag{5.50}$$

with $c = \lim_{n\to+\infty} c_n = -\operatorname{Log}\kappa = 1.76274\ldots$, and, up to $O(\frac{1}{n}\big(\big(\frac{\operatorname{Log}\operatorname{Log} n}{\operatorname{Log} n}\big)^2\big))$-terms,

$$\frac{(1-\frac{c_n}{n})^{2n}}{(1-\frac{c_n}{n}) - (1-\frac{c_n}{n})^n} = \frac{e^{-2c}}{1-e^{-c}}\Big(1 + \frac{c}{2n(1-e^{-c})}\big[2 - ce^{-c} - 2c\big]\Big) \tag{5.51}$$

with $e^{-2c}/(1-e^{-c}) = 0.0355344\ldots$

Proof. The asymptotic expansion (5.50) of c_n is deduced from the asymptotic expansions of ψ_n and $z_{J_n,n}$ given by (5.39) and (5.35) (Proposition 3.5 in [VG6]). We deduce the limit $c := -\operatorname{Log}(\kappa(1, a_{\max})) = 1.76274\ldots$ and then (5.51) follows. □

Definition 5.18. Let $n \geq n_2 := 260$. We denote by H_n the largest integer $j \geq \lceil v_n \rceil$ such that

$$\arg(z_{J_n,n}) - \arg(z_{j,n}) \geq \frac{(1-\frac{c_n}{n})^{2n}}{(1-\frac{c_n}{n}) - (1-\frac{c_n}{n})^n}. \tag{5.52}$$

Proposition 5.19. *Let $n \geq 260$. Let denote $\kappa := \kappa(1, a_{\max})$ for short. Then*

$$\arg(z_{H_n,n}) \;=\; 2\arcsin\big(\frac{\kappa}{2}\big) \;-\; \frac{\kappa^2}{1-\kappa}$$

$$+\frac{\operatorname{Log}\kappa}{n}\left[\frac{\kappa}{\sqrt{4-\kappa^2}} + \frac{2 + \kappa\operatorname{Log}(\kappa) + 2\operatorname{Log}(\kappa)}{2(1-\kappa)}\right] + \frac{1}{n} O\Big(\Big(\frac{\operatorname{Log}\operatorname{Log} n}{\operatorname{Log} n}\Big)^2\Big), \tag{5.53}$$

with, at the limit,

$$\lim_{n\to+\infty} \arg(z_{H_n,n}) = 2\arcsin\big(\frac{\kappa}{2}\big) \;-\; \frac{\kappa^2}{1-\kappa} = 0.13625.$$

Proof. The asymptotic expansion of the right-hand side term of (5.52) is

$$\frac{(1-\frac{c_n}{n})^{2n}}{(1-\frac{c_n}{n}) - (1-\frac{c_n}{n})^n} = \frac{e^{-2c}}{1-e^{-c}}\Big(1 + \frac{c(2 - ce^{-c} - 2c)}{2n(1-e^{-c})}\Big) + \ldots \tag{5.54}$$

Then the asymptotic expansion of $\arg(z_{H_n,n})$ comes from (5.52) in which the inequality is replaced by an equality, and from the asymptotic expansion (5.31) of $\arg(z_{J_n,n})$ (Proposition 5.12). □

For n large enough, $\arg(z_{H_n,n})$ is equal to $2\pi\frac{H_n}{n}$, up to higher order - terms, and a definition of H_n in terms of asymptotic expansions could be:

$$H_n = \lfloor \frac{n}{2\pi}\big(2\arcsin\big(\frac{\kappa}{2}\big) - \frac{\kappa^2}{1-\kappa}\big) - \mathrm{Log}\,(\kappa)\Big[\frac{\kappa}{\sqrt{4-\kappa^2}} + \frac{2+\kappa\,\mathrm{Log}\,(\kappa)+2\,\mathrm{Log}\,(\kappa)}{2(1-\kappa)}\Big]\rfloor, \tag{5.55}$$

For simplicity's sake, we will take the following definition of H_n

$$H_n := \lfloor \frac{n}{2\pi}\big(2\arcsin\big(\frac{\kappa}{2}\big) \ - \ \frac{\kappa^2}{1-\kappa}\big) - 1\rfloor. \tag{5.56}$$

Remark 5.20. The value $n_2 = 260$ is calculated by the inequality $\frac{2\pi v_n}{n} < \arg(z_{H_n,n})$ which should be valid for all $n \geq 260$, where H_n is given by (5.56), $\arg(z_{H_n,n})$ by (5.53), where (v_n) is the delimiting sequence (cf Appendix) of the transition region of the boundary of the bump sector. A first minimal value of n is first estimated by $2\pi\frac{\mathrm{Log}\,n}{n} < D(\arg(z_{H_n,n}))$ using (5.53). Then it is corrected so that the numerical value of the tail of the asymptotic expansion in (5.53) be taken into account in this inequality.

Theorem 5.21. *Let $n \geq n_2 := 260$. Denote by $\mathscr{D}_n$ the subdomain of the open unit disc, symmetrical with respect to the real axis, defined by the conditions:*

$$|z| < 1 - \frac{c_n}{n}, \qquad \frac{1}{n}\left(\frac{\mathrm{Log}\,\mathrm{Log}\,n}{\mathrm{Log}\,n}\right)^2 < |z-\theta_n|, \tag{5.57}$$

$$\frac{\pi|z_{j,n}|}{n\,a_{\max}} < |z - z_{j,n}|, \qquad \textit{for } \ j = 1,2,\ldots,J_n, \tag{5.58}$$

and, for $j = J_n+1,\ldots,2J_n - H_n + 1$,

$$\frac{\pi|z_{j,n}|}{n\,s_{j,n}} < |z - z_{j,n}|, \qquad \textit{with } \ s_{j,n} = a_{\max}\Big[1 + \frac{a_{\max}^2 (j-J_n)^2}{\pi^2 J_n^2}\Big]^{-1/2}. \tag{5.59}$$

Then, for any real number $\beta > 1$ having $\mathrm{dyg}(\beta) = n$, the Parry Upper function $f_\beta(z)$ does not vanish at any point z in $\mathscr{D}_n$.

Proof. Assume $\beta > 1$ such that $\theta_{n-1} < \beta^{-1} < \theta_n$. We will apply the general form of the Theorem of Rouché to the compact $\mathscr{K}_n$ which is the adherence of the domain $\mathscr{D}_n$, i.e. we will show that the inequality (and symmetrically with respect to the real axis)

$$|f_\beta(z) - G_n(z)| < |G_n(z)|, \qquad z \in \partial\mathscr{K}_n^{ext} \cup C_{1,n} \cup C_{2,n} \cup \ldots \cup C_{J_n,n} \tag{5.60}$$

holds, with $z \in \mathrm{Im}(z) \geq 0$, where $\partial\mathscr{K}_n$ is the union of: (i) the arcs of the circles defined by the equalities in (5.58) and (5.59), arcs which lie in $|z| \leq 1 - c_n/n$, and circles for which the intersection with $|z| = 1 - c_n/n$ is not empty, (ii) the arcs of $C(0, 1-c_n/n)$ which have empty intersections with the interiors of the discs defined by the inequalities ">", instead of "<", in (5.58) and (5.59), which join two successive circles. The

two functions $f_\beta(z)$ and $G_n(z)$ are continuous on the compact $\mathscr{K}_n$, holomorphic in its interior $\mathscr{D}_n$, and G_n has no zero in $\mathscr{K}_n$. As a consequence the function $f_\beta(z)$ will have no zero in the interior $\mathscr{D}_n$ of $\mathscr{K}_n$.

Instead of using $f_\beta(z)$ itself in (5.60), we will show that the following inequality holds true

$$\frac{|z|^{2n-1}}{1-|z|^{n-1}} \;<\; |-1+z+z^n|, \qquad \text{for all } z \in \partial\mathscr{K}_n^{ext} \tag{5.61}$$

what will imply the claim.

The Rouché inequalities (5.60) (5.61) hold true on the (complete) circles $C_{j,n}, 1 \leq j \leq J_n$ by Theorem 5.9 and Proposition 5.14; these conditions become out of reach for j taking higher values (i.e. in $\{J_n+1,\ldots,\lfloor n/6 \rfloor\}$), but we will show that they remain true on the arcs defined by the equalities in (5.59). The domain $\mathscr{D}_n$ only depends upon the dynamical degree n of β, not of β itself.

Let us prove that the external Rouché circle $|z| = 1-c_n/n$ intersects all the circles $C_{J_n-k,n}, k = 0,1,\ldots,k_{\max}$, with $k_{\max} := \lfloor J_n(\frac{\pi}{a_{\max}}) \rfloor$. Indeed, up to $\frac{1}{n}O\big(\big(\frac{\mathrm{Log}\,\mathrm{Log}\,n}{\mathrm{Log}\,n}\big)^2\big)$-terms, from Proposition 5.12,

$$\mathrm{Log}\,(2\sin(\pi\frac{J_n}{n})) = \mathrm{Log}\,(2\sin(\pi\frac{(J_n-k)+k}{n})) = \mathrm{Log}\,\big(2\pi\frac{J_n-k}{n}(1+\frac{k}{J_n-k})\big)$$

$$= \mathrm{Log}\,\big(2\sin(\pi\frac{J_n-k}{n})\big) + \frac{k}{J_n}. \tag{5.62}$$

Since $|z_{J_n,n}| = 1-c_n/n = 1+\frac{1}{n}\mathrm{Log}\,(2\sin(\pi\frac{J_n}{n}))+\frac{1}{n}O\big(\big(\frac{\mathrm{Log}\,\mathrm{Log}\,n}{\mathrm{Log}\,n}\big)^2\big)$, we deduce from (5.62), with $k \leq k_{\max}$, that the point $z \in C(0,1-c_n/n)$ for which $\arg(z) = \arg(z_{J_n-k,n})$ is such that

$$|z_{J_n-k,n} - z| \;=\; \frac{k}{nJ_n} \;\leq\; \frac{\lfloor J_n(\frac{\pi}{a_{\max}}) \rfloor}{nJ_n} \;\leq\; \frac{\pi}{n\,a_{\max}}$$

up to $\frac{1}{n}O\big(\big(\frac{\mathrm{Log}\,\mathrm{Log}\,n}{\mathrm{Log}\,n}\big)^2\big)$- terms. As soon as n is large enough, we deduce that z lies in the interior of $D_{J_n-k,n}$. Since the function $x \to \mathrm{Log}\,(2\sin(\pi x))$ is negative and strictly increasing on $(0,1/6)$, the sequence $(|z_{j,n}|)_{j=H_n,\ldots,J_n}$ is strictly increasing, by (5.35). Hence we deduce that the circle $|z| = 1-c_n/n$ intersects all the circles $C_{j,n}$ for $j = J_n - k_{\max},\ldots,J_n$.

The same arguments show that the external Rouché circle $|z| = 1-c_n/n$ intersects all the circles $C(z_{j,n}, \frac{\pi|z_{j,n}|}{n s_{j,n}})$ for $j = J_n+1, J_n+2,\ldots,2J_n-H_n+1$.

The quantities $s_{j,n}$, for $j = J_n+1,\ldots,2J_n-H_n+1$, are easily calculated (left to the reader) so that the distance (length of the j-th circle segment)

$$\left|\; \frac{z_{j,n}}{|z_{j,n}|}(1-\frac{c_n}{n}) - y_j \;\right| = \left|\; \frac{z_{j,n}}{|z_{j,n}|}(1-\frac{c_n}{n}) - y'_j \;\right|$$

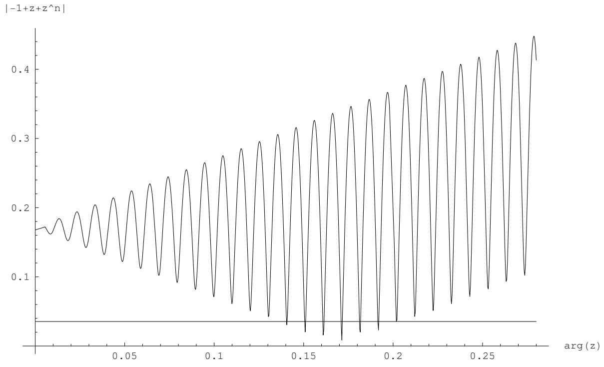

Figure 3. Oscillations of the upper bound $|-1+z+z^n|$ of the Rouché inequality (5.20), for z running over the curve $|z| = 1-c_n/n$ (here represented with $n = 615$) as a function of $\arg(z)$ in $[0,0.28]$. The minima correspond to the angular positions of the zeroes $z_{j,n}$ of the trinomial $-1+X+X^n$, for $j = 1,2,\ldots,H_n,\ldots,J_n,\ldots,2J_n-H_n+1,\ldots$ ($J_{615} = 17, H_{615} = 12$). The angular separation between two successive minima is $\approx 2\pi/n$. The difference between two successive minima is $\approx 2\pi/n$. For $n = 615$, the arguments $2\pi(\operatorname{Log} n)/n$ (limiting the bump sector), $\arg(z_{H_n,n})$ and $\arg(z_{J_n,n})$ are respectively equal to $0.0656\ldots, 0.12189\ldots, 0.17129\ldots$. The horizontal line at the y-coordinate 0.0354... is the value of the left-hand side term of the Rouché inequality (5.20) (Proposition 5.60); it is always strictly smaller than the minimal value of the oscillating function $|-1+z+z^n|$ on the external boundary $\partial\mathscr{K}_n^{ext}$, whose geometry surrounds the roots $z_{j,n}$ for j between H_n+1 and $2J_n-H_n+1$.

for $y_j, y'_j \in C(z_{j,n}, \frac{\pi|z_{j,n}|}{n s_{j,n}}) \cap C(0, 1-\frac{c_n}{n}), y_j \neq y'_j$, be independent of j in the interval $\{J_n+1,\ldots,2J_n-H_n+1\}$ and equal to

$$\frac{\pi|z_{J_n,n}|}{n\, a_{\max}}. \tag{5.63}$$

Then the two sequences of moduli of centers $(|z_{j,n}|)_{j=J_n+1,\ldots,2J_n-H_n+1}$ and of radii $(\frac{\pi|z_{j,n}|}{n s_{j,n}})_{j=J_n+1,\ldots,2J_n-H_n+1}$ are both increasing, with the fact that the corresponding discs $D(z_{j,n}, \frac{\pi|z_{j,n}|}{n s_{j,n}})$ keep constant the intersection chord $\arg(y_j) - \arg(y'_j) = \frac{\pi|z_{J_n,n}|}{n\, a_{\max}}$ with the external Rouché circle $|z| = 1-c_n/n$.

Let $z \in C(0,1-\frac{c_n}{n})$, $\varphi := \arg(z) \in [0,\pi]$. Denote by $Z(\varphi) := |G_n((1-\frac{c_n}{n})e^{i\varphi})|^2 = \left|-1+(1-\frac{c_n}{n})e^{i\varphi}+(1-\frac{c_n}{n})^n e^{in\varphi}\right|^2$. The expansion of the function $Z(\varphi)$ as a function of φ, up to $O(1/n)$- terms, is the following: $Z(\varphi) =$

$$(-1+(1-\frac{c_n}{n})\cos(\varphi)+(1-\frac{c_n}{n})^n\cos(n\varphi))^2+((1-\frac{c_n}{n})\sin(\varphi))+(1-\frac{c_n}{n})^n\sin(n\varphi))^2$$

$$= 2 + e^{-2c} - 2\cos(\varphi) - 2e^{-c}\cos(n\,\varphi) + 2e^{-c}\cos(\varphi)\cos(n\,\varphi) + 2e^{-c}\sin(\varphi)\sin(n\,\varphi)$$

$$= 2 + e^{-2c} - 2\cos(\varphi) - 4e^{-c}\sin(\frac{\varphi}{2})\Big(\cos(n\,\varphi)\sin(\frac{\varphi}{2}) - \sin(n\,\varphi)\cos(\frac{\varphi}{2})\Big)$$

$$= 2 + e^{-2c} - 2\cos(\varphi) + 4e^{-c}\sin(\frac{\varphi}{2})\sin(n\varphi - \frac{\varphi}{2}). \tag{5.64}$$

The function $Z(\varphi)$, defined on $[0,\pi/3]$, is almost-periodic (in the sense of Besicovitch and Bohr), takes the value 0 at $\varphi = \arg(z_{J_n,n})$, and therefore, up to $O(1/n)$-terms, has its minima at the successive arguments $\arg(z_{J_n,n}) + \frac{2k\pi}{n}$ for $|k| = 0,1,2,\ldots,J_n - H_n + 1,\ldots$ (Figure 3). For such integers k, from (5.64), we deduce the successive minima

$$|-1 + z_{J_n,n}e^{-2ik\pi/n} + (z_{J_n,n}e^{-2ik\pi/n})^n| = |G_n(z_{J_n,n})| + \frac{2|k|\pi}{n} = \frac{2|k|\pi}{n} \tag{5.65}$$

up to $\frac{1}{n}O\big(\big(\frac{\mathrm{Log}\,\mathrm{Log}\,n}{\mathrm{Log}\,n}\big)^2\big)$- terms, with $\arg(z_{J_n,n}e^{-2ik\pi/n}) = \arg(z_{J_n-k,n})$ up to $O(1/n)$-terms.

With the above notations, denote by y_j, y'_j the two points of $C(0,1-\frac{c_n}{n})$ which belong to $C_{j,n}$ for $2H_n - J_n \leq j \leq J_n$, to $C(z_{j,n},\frac{\pi|z_{j,n}|}{n s_{j,n}})$ for $J_n + 1 \leq j \leq 2J_n - H_n + 1$. Writing them by increasing argument, we have:

$$y_{2H_n-J_n}, y'_{2H_n-J_n}, \ldots, y_{H_n}, y'_{H_n}, \ldots, y_{J_n}, y'_{J_n},\ y_{J_n+1}, y'_{J_n+1}, \ldots, y_{2J_n-H_n+1},\ y'_{2J_n-H_n+1}. \tag{5.66}$$

The Rouché inequality (5.60) is obviously satisfied at each y_j and y'_j for $j = 2H_n - J_n,\ldots,J_n$. Let us show that this inequality holds at each point y_j and y'_j for $j = J_n+1,\ldots,2J_n-H_n+1$. Indeed, for such a point, say y_j, there exists

$$\xi_j = w_j\, z_{J_n,n}e^{2i(j-J_n)\pi/n} + (1-w_j)\,y_j, \quad \text{for some } w_j \in [0,1],$$

lying in the segment $\Big[z_{J_n,n}e^{2i(j-J_n)\pi/n}, y_j\Big]$ such that

$$G_n(y_j) = G_n(z_{J_n,n}e^{2i(j-J_n)\pi/n}) + (y_j - z_{J_n,n}e^{2i(j-J_n)\pi/n})\,G'_n(\xi_j)$$

with, using (5.63),

$$|G_n(y_j) - G_n(z_{J_n,n}e^{2i(j-J_n)\pi/n})| = |y_j - z_{J_n,n}e^{2i(j-J_n)\pi/n}||G'_n(\xi_j)| = \frac{\pi|z_{J_n,n}|}{n\,a_{\max}}|G'_n(\xi_j)|.$$

The derivative of $G_n(z)$ is $G'_n(z) = 1 + nz^{n-1}$. Up to $O(1/n)$-terms, the line generated by the segment $\Big[z_{J_n,n}e^{2i(j-J_n)\pi/n}, y_j\Big]$ is tangent to the circle $C(0,1-c_n/n)$, and the modulus $\frac{1}{n}|G'_n(\xi_j)|$ satisfies

$$\frac{1}{n}|G'_n(\xi_j)| = \frac{1}{n}|G'_n(z_{J_n,n}e^{2i(j-J_n)\pi/n})| = \frac{1}{n}|G'_n(z_{J_n,n})| = \lim_{n\to+\infty}\frac{1}{n}|G'_n(z_{J_n,n})| = e^{-c}.$$

From $|G_n(y_j)| \geq \big||G_n(y_j) - G_n(z_{J_n,n} e^{2i(j-J_n)\pi/n})| - |G_n(z_{J_n,n} e^{2i(j-J_n)\pi/n})|\big|$ and (5.63) we deduce

$$|G_n(y_j)| \geq \frac{\pi |z_{J_n,n}|}{a_{\max}} e^{-c} - \frac{2\pi |j - J_n|}{n}. \tag{5.67}$$

But, by definition of H_n, still up to $O(1/n)$-terms, for $|j - J_n| \leq J_n - H_n - 1$,

$$\frac{2\pi |j - J_n|}{n} \leq \frac{2\pi\,(J_n - H_n - 1)}{n} = \arg(z_{J_n,n}) - \arg(z_{H_n+1,n}) \leq \frac{e^{-2c}}{1 - e^{-c}}. \tag{5.68}$$

This inequality is in particular satisfied for the last two values of $|j - J_n|$ which are $J_n - H_n$ and $J_n - H_n + 1$ up to $O(1/n)$-terms. Since the inequality

$$0.0710\ldots = \ 2\frac{e^{-2c}}{1 - e^{-c}} \ < \ \frac{\pi |z_{J_n,n}|}{a_{\max}} e^{-c} \ = 0.0914\ldots \tag{5.69}$$

holds, from (5.67), (5.68) and (5.69), as soon as n is large enough, we deduce the Rouché inequality

$$|G_n(y_j)| \ \geq \ \frac{\pi |z_{J_n,n}|}{a_{\max}} - \frac{e^{-2c}}{1 - e^{-c}} \ \geq \ \frac{e^{-2c}}{1 - e^{-c}}.$$

Therefore the conditions of Rouché (5.61) hold at all the points y_j and y'_j of (5.66).

Let us prove that the conditions of Rouché (5.61) hold on each arc $y'_j\, y_{j+1}$ of the circle $|z| = 1 - c_n/n$, for $j = 2H_n - J_n, 2H_n - J_n + 1, \ldots, 2J_n - H_n$. Indeed, from (5.64), the derivative $Z'(\varphi)$ takes a positive value at the extremity y'_j while it takes a negative value at the other extremity y_{j+1}. $Z(\varphi)$ is almost-periodic of almost-period $2\pi/n$. The function $\sqrt{Z(\varphi)}$ is increasing on $(\arg(z_{j,n}), \arg(z_{j,n}) + \frac{\pi}{n})$ and decreasing on $(\arg(z_{j,n}) + \frac{\pi}{n}, \arg(z_{j,n}) + 2\frac{\pi}{n})$; on the arc $y'_j\, y_{j+1}$ it takes the value $|G_n(y'_j)| \geq \frac{e^{-2c}}{1-e^{-c}}$, admits a maximum, and decreases to $|G_n(y_{j+1})| \geq \frac{e^{-2c}}{1-e^{-c}}$. Hence, (5.20) holds true for all $z \in C(0, 1 - c_n/n)$ with $\arg(y'_j) \leq \arg(z) \leq \arg(y_{j+1})$.

Let us now prove that the condition of Rouché (5.20) is satisfied in the angular sector $0 \leq \arg(z) \leq \arg(z_{H_n,n})$. Indeed, in this angular sector, the successive minima of $\sqrt{Z(\varphi)}$ are all above $\frac{e^{-2c}}{1-e^{-c}}$ by the definition of H_n and (5.65). Hence the claim.

Let us prove that the condition of Rouché (5.20) is satisfied in the angular sector $\arg(z_{2J_n - H_n + 1,n}) \leq \arg(z) \leq \frac{\pi}{2}$. In this angular sector, the oscillations of $\sqrt{Z(\varphi)}$ still occur by the form of (5.64) and the successive minima of $\sqrt{Z(\varphi)}$ are all above $\frac{e^{-2c}}{1-e^{-c}}$ for $\frac{2J_n - H_n + 2}{J_n} \leq \arg(z) \leq \pi/2$, by (5.65) for $k \geq J_n - H_n + 1$. We deduce the claim.

The condition of Rouché (5.20) is also satisfied in the angular sector $\pi \leq \arg(z) \leq \pi/2$, since then $\cos(\varphi) \leq 0$ and therefore $\sqrt{Z(\varphi)} \geq \sqrt{2 + e^{-2c} - 4e^{-c}} = 1.15\ldots$. Since this lower bound is greater than the value $\frac{e^{-2c}}{1-e^{-c}} = 0.0354\ldots$ we deduce the claim.

Let us show that the conditions of Rouché (5.20) are also satisfied on the arcs $C(z_{j,n}, \frac{\pi |z_{j,n}|}{n s_{j,n}}) \cap \overline{D}(0, 1 - \frac{c_n}{n})$ for $j = J_n + 1, \ldots, 2J_n - H_n + 1$. For such an integer

j, let us denote such an arc by $y_j\ y'_j$. The two extremities y_j and y'_j of the arc $y_j\ y'_j$ of the circle $C(z_{j,n}, \frac{\pi|z_{j,n}|}{n s_{j,n}})$ define the same value of the difference cosine, say $X_j := \cos(\arg(y_j - z_{j,n}) - \arg(z_{j,n})) = \cos(\arg(y'_j - z_{j,n}) - \arg(z_{j,n}))$, by (5.63). The conditions of Rouché are already satisfied at the points y_j and y'_j by the above. Recall that, for any fixed $a \geq 1$, the function $\kappa(X,a)$, defined in (5.27), is such that the partial derivative $\partial\kappa_X$ of $\kappa(X,a)$ is strictly negative on the interior of $[-1,+1]\times[1,+\infty)$. In particular the function $\kappa(X, s_{j,n})$ is decreasing. For any point Ω of the arc $y_j\ y'_j$, we denote by $X = \cos(\arg(\Omega - z_{j,n}) - \arg(z_{j,n}))$. We deduce, up to $O(1/n)$-terms,

$$\frac{e^{-2c}}{1-e^{-c}} \ \leq\ \kappa(X_j, s_{j,n}) \ \leq\ \kappa(X, s_{j,n}), \qquad \text{for all } \ X \in [-1, X_j],$$

hence the result. □

Remark 5.22. In the case where $\beta \in (1, \theta_6^{-1})$ is an algebraic integer such that $\beta \notin \{\theta_n^{-1} \mid n \geq 6\}$, the lenticulus $\mathscr{L}_\beta$ of Galois conjugates of $1/\beta$ in the angular sector $\arg z \in \{-\frac{\pi}{3}, +\frac{\pi}{3}\}$ is obtained by truncation and a slight deformation of $\mathscr{L}_{\theta^{-1}_{\mathrm{dyg}(\beta)}}$. The asymptotic expansion of the minorant of the Mahler measure $\mathrm{M}(\beta)$ will be obtained from this lenticulus as a function of the dynamical degree $\mathrm{dyg}(\beta)$.

5.4 Minoration of the Mahler measure: a continuous lower bound

The passage from the zeroes $\omega_{j,n}$ of the Parry Upper function $f_\beta(z)$ to the zeroes of the minimal polynomial of β is crucial. It is crucial since the Mahler measure $\mathrm{M}(\beta)$ is constructed from the roots of $P_\beta(z)$, by its very definition, and not from the roots of the analytic function $f_\beta(z)$ or the poles of $\zeta_\beta(z)$.

The key result which makes the link between $\mathrm{M}(\beta)$ and $f_\beta(z)$ is Theorem 5.23. Theorem 5.23 extends Theorem 5.5; it gives the extension of the domain where the minimal polynomial (function) $P_\beta(z)$ is fracturable and for which the power series $U_\beta(z)$ in its decomposition is holomorphic, with nonvanishing properties on the lenticulus of zeroes of $f_\beta(z)$. This extension allows the identification of the lenticulus of zeroes $\omega_{j,n}$ of $f_\beta(z)$ as lenticulus of conjugates of β. The complete set of conjugates of β is certainly out of reach by this method, and the domain of holomorphy of $U_\beta(z)$ is very probably larger. It only gives a subproduct of the product defining $\mathrm{M}(\beta)$, hence a minoration of $\mathrm{M}(\beta)$.

Theorem 5.23. *Let $n \geq 260$ and $\beta > 1$ any algebraic integer such that* $\mathrm{dyg}(\beta) = n$. *Let $\mathscr{D}_n$ be the subdomain of the open unit disc defined in Theorem 5.21. Denote $D_{j,n} := \{z \mid |z - z_{j,n}| < \frac{\pi|z_{j,n}|}{n a_{\max}}\}$, $j = 1,2,\ldots,J_n$. Then the domain of holomorphy of the analytic function $U_\beta(z) = \frac{P_\beta(z)}{f_\beta(z)} \in \mathbb{Z}[[z]]$, quotient of the minimal polynomial of β*

by the Parry Upper function at β, contains the connected domain

$$\Omega_n := \mathscr{D}_n \cup \bigcup_{j=1}^{J_n} \left(D_{j,n} \cup \overline{D_{j,n}}\right) \cup D(\theta_n, \frac{t_{0,n}}{n}),$$

itself containing the open disc $D(0, 1 - \frac{c_n}{n} - \frac{\pi |z_{J_n,n}|}{n\, a_{\max}})$. The fracturability of the minimal polynomial function $P_\beta(z)$ of β, in $\mathbb{Z}[[z]]$, obeys the Carlson-Polya dichotomy as in Theorem 5.5, admitting the following factorization

$$P_\beta(z) = U_\beta(z) \times f_\beta(z)$$

where $U_\beta(z)$ does not vanish on the lenticulus $\mathscr{L}_\beta = \{\frac{1}{\beta}\} \cup \bigcup_{j=1}^{J_n} (\{\omega_{j,n}\} \cup \{\overline{\omega_{j,n}}\}) \subset \Omega_n$. For any zero $\omega_{j,n} \in \mathscr{L}_\beta$, and symmetrically by complex conjugation with respect to the real axis, we have

$$U_\beta(\omega_{j,n}) \;=\; \frac{P'_\beta(\omega_{j,n})}{f'_\beta(\omega_{j,n})} \neq 0 \quad \textit{and} \quad U_\beta(\frac{1}{\beta}) \;=\; \frac{P'_\beta(\frac{1}{\beta})}{f'_\beta(\frac{1}{\beta})} \neq 0. \qquad (5.70)$$

All the zeroes of the lenticulus $\mathscr{L}_\beta$ of $f_\beta(z)$ are zeroes of the minimal polynomial $P_\beta(z)$ of β.

Proof. From Theorem 5.5, a lower bound of the radius of convergence $U_\beta(z)$ is $1 - \frac{\mathrm{Log}\,(n-1)}{n-1}$. Using Theorem 5.21 we will show that this lower bound can be improved to $1 - \frac{c_n}{n} - \frac{\pi |z_{J_n,n}|}{n\, a_{\max}}$, and that the domain of holomorphy of $U_\beta(z)$ even extends to a collection of portions of Rouché discs centered at roots $z_{j,n}$ of G_n, given by Ω_n.

For β a Parry number or not, the analytic function $f_\beta(z)$ is holomorphic on $|z| < 1$ by Theorem 4.11 and Theorem 4.12 and the unit circle is not the natural boundary of $f_\beta(z)$ or is, accordingly. Consequently the function $U_\beta(z) = P_\beta(z)/f_\beta(z)$ is analytic on the open unit disc and its poles are identified from the zeroes ω of $f_\beta(z)$: (i) the multiple poles (multiplicity ≥ 2) come from the zeroes of f_β of higher multiplicity, in $|z| < 1$, since the roots of $P_\beta(z)$ are all of multiplicity one, (ii) the simple poles come from simple zeroes of $f_\beta(z)$ except if these zeroes ω are simultaneously of multiplicity one and zeroes of the minimal polynomial $P_\beta(z)$. We will show that, with all the zeroes of $\mathscr{L}_\beta$, we are in the exception of case (ii).

The characterization of the subdomains of $D(0,1)$ on which the function $f_\beta(z)$ has no zero, resp. a simple zero, is given in Theorem 5.21 and Lemma 5.8: $\mathscr{D}_n$, resp. $D_{j,n}$ and $\overline{D_{j,n}}$ for each $j = 1, 2, \ldots, J_n$, symmetrically, and $D(\theta_n, \frac{t_{0,n}}{n})$. Therefore $U_\beta(z)$ has no pole in $\mathscr{D}_n$. Let us observe that all the arcs $y_j\, y'_j$, $j \geq J_n + 1$, given by (5.59) in Theorem 5.21, lie outside the open disc $D(0, 1 - \frac{c_n}{n} - \frac{\pi |z_{J_n,n}|}{n\, a_{\max}})$. Now, for every $j \in \{1, 2, \ldots, J_n\}$ such that $D_{j,n}$ intersects the circle $|z| = 1 - \frac{c_n}{n} - \frac{\pi |z_{J_n,n}|}{n\, a_{\max}}$ the unique zero $\omega_{j,n}$ of $f_\beta(z)$ which belongs to $D_{j,n}$ either belongs to $D(0, 1 - \frac{c_n}{n} - \frac{\pi |z_{J_n,n}|}{n\, a_{\max}})$ or not. If $\omega_{j,n} \notin D(0, 1 - \frac{c_n}{n} - \frac{\pi |z_{J_n,n}|}{n\, a_{\max}})$, the function $U_\beta(z)$ has no zero and no pole in

$D(0, 1 - \frac{c_n}{n} - \frac{\pi |z_{J_n,n}|}{n\, a_{\max}}) \cap D_{j,n}$. For those j for which $\omega_{j,n} \in D(0, 1 - \frac{c_n}{n} - \frac{\pi |z_{J_n,n}|}{n\, a_{\max}}) \cap D_{j,n}$, the derivation of the formal identity: $P_\beta(X) = U_\beta(X) \times f_\beta(X)$ gives:

$$P'_\beta(X) \;=\; U'_\beta(X)\, f_\beta(X) \;+\; U_\beta(X)\, f'_\beta(X). \tag{5.71}$$

Since $\omega_{j,n}$ is a simple zero, $f'_\beta(\omega_{j,n}) \neq 0$. Specializing the formal variable X to $\omega_{j,n}$ we obtain:

$$P'_\beta(\omega_{j,n}) \;=\; U_\beta(\omega_{j,n})\, f'_\beta(\omega_{j,n}),$$

that is (5.70) with $|U_\alpha(\omega)| \neq +\infty$, meaning that $\omega_{j,n}$ is not a pole of $U_\beta(z)$. The zeroes of $\mathscr{L}_\beta$ are not singularities of $U_\beta(z)$. Let us show that $U_\beta(\omega_{j,n}) \neq 0$. Let us assume the contrary. Since $P_\beta(X)$ has simple roots, the value $P_\beta(\omega_{j,n})$ is either nonzero or zero with multiplicity one. But, from the relation $P_\beta(z) = U_\beta(z) \times f_\beta(z)$, it would imply that the multiplicity of the zero $z = \omega_{j,n}$ of $P_\beta(z)$ is ≥ 2. Contradiction.

With the same arguments, for every $j \in \{1, 2, \ldots, H_n\}$ such that $D_{j,n} \subset D(0, 1 - \frac{c_n}{n} - \frac{\pi |z_{J_n,n}|}{n\, a_{\max}})$, the (simple) zeroes $\omega_{j,n}$ of the Parry Upper function $f_\beta(z)$ are such that $|U_\beta(1/\beta)| \neq 0, \neq +\infty$. In a similar way, the (simple) zero $1/\beta$ of $P_\beta(z)$ is a (simple) zero of the Parry Upper function $f_\beta(z)$ at β for which $|U_\beta(1/\beta)| \neq 0, \neq \infty$.

We deduce that all the lenticular zeroes of $\mathscr{L}_\beta$ are zeroes of the minimal polynomial $P_\beta(X)$ of β. □

Let $n \geq 260$ and $\beta > 1$ be an algebraic integer such that $\theta_{n-1}^{-1} < \beta < \theta_n^{-1}$. The minimal polynomial of β is factorized under the form:

$$P_\beta(z) \;=\; \prod_{\omega \in \mathscr{L}_\beta} (z - \omega) \;\times\; \prod_{\omega \notin \mathscr{L}_\beta} (z - \omega). \tag{5.72}$$

Flatto, Lagarias and Poonen [FLP] have shown and studied the continuity of the modulus of the second smallest root of $f_\beta(z)$ as a function of β. Theorem 5.24 extends this result.

Theorem 5.24. *Let $n \geq 260$. Let $\beta > 1$ be an algebraic integer such that* $\mathrm{dyg}(\beta) = n$*. The product, called lenticular Mahler measure of β, defined by*

$$\mathrm{M}_r(\beta) \;:=\; \prod_{\omega \in \mathscr{L}_\beta} |\omega|^{-1} \tag{5.73}$$

is a continuous function of β on the open interval $(\theta_n^{-1}, \theta_{n-1}^{-1})$, which admits the following left and right limits

$$\lim_{\beta \to \theta_{n-1}^{-1\,-}} \mathrm{M}_r(\beta) = \prod_{\omega \in \mathscr{L}_{\theta_{n-1}^{-1}}} |\omega|^{-1} = \theta_{n-1}^{-1} \times \prod_{\substack{1 \leq j \leq J_n \\ z_{j,n-1} \in \mathscr{L}_{\theta_{n-1}^{-1}}}} |z_{j,n-1}|^{-2}, \tag{5.74}$$

$$\lim_{\beta \to \theta_n^{-1\,+}} \mathrm{M}_r(\beta) = \prod_{\omega \in \mathscr{L}_{\theta_n^{-1}}} |\omega|^{-1} = \theta_n^{-1} \times \prod_{\substack{1 \leq j \leq J_n \\ z_{j,n} \in \mathscr{L}_{\theta_n^{-1}}}} |z_{j,n}|^{-2}. \tag{5.75}$$

The discontinuity (jump) of $\mathrm{M}_r(\beta)$ *at the Perron number* θ_{n-1}^{-1}*, given in the multiplicative form by*

$$\frac{\lim_{\beta \to \theta_{n-1}^{-1}-} \mathrm{M}_r(\beta)}{\lim_{\beta \to \theta_{n-1}^{-1}+} \mathrm{M}_r(\beta)} \;=\; |z_{J_n,n-1}|^{-2}, \tag{5.76}$$

tends to 1 (i.e. disappears at infinity) when $n = \mathrm{dyg}(\beta)$ *tends to infinity.*

Proof. From Corollary 4.25 in §4.4 all the maps $\beta \to \omega(\beta) \in \mathscr{L}_\beta$ are continuous. Now the identification of the zeroes of the Parry Upper function $f_\beta(z)$ as conjugates of β, from Theorem 5.23, allows to consider this continuity property as a continuity property over the conjugates of β which define the lenticulus $\mathscr{L}_\beta$. As a consequence all the maps $\beta \to |\omega(\beta)| \in \mathscr{L}_\beta$, are continuous, as well as their product (5.73).

Let $1 \leq j \leq J_n$. Let us prove that $z_{j,n-1} \in D_{j,n} = \{z \mid |z - z_{j,n}| < \frac{\pi |z_{j,n}|}{n\, a_{\max}}\}$. Indeed,

$$|z_{j,n}| = 1 + \frac{1}{n} \mathrm{Log}\,(2 \sin(\frac{\pi j}{n})) + \ldots, \qquad \arg(z_{j,n}) = \ldots$$

and

$$|z_{j,n-1}| = 1 + \frac{1}{n-1} \mathrm{Log}\,(2 \sin(\frac{\pi j}{n-1})) + \ldots, \qquad \arg(z_{j,n-1}) = \ldots$$

so that, easily,

$$|z_{j,n} - z_{j,n-1}| < \frac{\pi |z_{j,n}|}{n\, a_{\max}}. \tag{5.77}$$

The image of the interval $(\theta_n^{-1}, \theta_{n-1}^{-1}) \cap \mathscr{O}_{\overline{\mathbb{Q}}}$ by a map $\beta \to \omega_{j,n}(\beta) \in \mathscr{L}_\beta$ is a curve in $D_{j,n}$ over $\mathscr{O}_{\overline{\mathbb{Q}}}$ with extremities $z_{j,n}$ and $z_{j,n-1}$, both in $D_{j,n}$ by (5.77). This curve does not intersect itself. Indeed, if it would be a self-intersecting curve we would have, for two distinct algebraic integers β and β', the same conjugate in $D_{j,n}$, what is impossible since P_β and P'_β are both irreducible, and therefore they cannot have a root in common. This curve does not ramify either by the uniqueness property imposed locally by the Theorem of Rouché. We deduce the left limit (5.74) and the right limit (5.75) by continuity. $\square$

The subscript "r" added to the "M" of the Mahler measure stands for "reduced to the lenticulus".

Remark 5.25. Decomposing the Mahler measure gives

$$\mathrm{M}(\beta) = \prod_{\omega \in \mathscr{L}_\beta} |\omega|^{-1} \times \prod_{\substack{\omega \notin \mathscr{L}_\beta, |\omega|<1 \\ P_\beta(\omega)=0}} |\omega|^{-1}.$$

Theorem 5.24, for which the Rouché method has been applied, shows the continuity of the partial product $\beta \to \prod_{\omega \in \mathscr{L}_\beta} |\omega|^{-1}$, associated with the identified lenticulus of

conjugates of β, with β running over each open interval of extremities two successive Perron numbers θ_n^{-1}. It is very probable that a method finer than the method of Rouché would lead to a higher value of J_n, to more zeroes of $f_\beta(z)$ identified as conjugates of β, and the disappearance of the discontinuities (jumps) in (5.76).

Theorem 5.26. *Let $\beta > 1$ be an algebraic integer such that $\mathrm{dyg}(\beta) \geq 260$. Denote $\kappa = \kappa(1, a_{\max})$. The Mahler measure $\mathrm{M}(\beta)$ is bounded from below by the lenticular Mahler measure of β as*

$$\mathrm{M}(\beta) = \mathrm{M}_r(\beta) \times \prod_{\substack{\omega \notin \mathscr{L}_\beta, |\omega|<1 \\ P_\beta(\omega)=0}} |\omega|^{-1} \geq \mathrm{M}_r(\beta).$$

Denoting

$$\Lambda_r := \exp\Big(\frac{-1}{\pi} \int_0^{2\arcsin(\frac{\kappa}{2})} \mathrm{Log}\,\big(2\sin(\frac{x}{2})\big) dx\Big) = 1.16302\ldots,$$

and

$$\mu_r := \exp\Big(\frac{-1}{\pi} \int_0^{2\arcsin(\frac{\kappa}{2})} \mathrm{Log}\,\Big[\frac{1+2\sin(\frac{x}{2}) - \sqrt{1-12\sin(\frac{x}{2})+4(\sin(\frac{x}{2}))^2}}{8\sin(\frac{x}{2})}\Big] dx\Big)$$

$$= 0.992337\ldots,$$

the lenticular Mahler measure $\mathrm{M}_r(\beta)$ of β admits a liminf and a limsup when β tends to 1^+, equivalently when $\mathrm{dyg}(\beta)$ tends to infinity, respectively bounded from below and above as

$$\liminf_{\mathrm{dyg}(\beta)\to+\infty} \mathrm{M}_r(\beta) \geq \Lambda_r \cdot \mu_r = 1.15411\ldots, \tag{5.78}$$

$$\limsup_{\mathrm{dyg}(\beta)\to+\infty} \mathrm{M}_r(\beta) \leq \Lambda_r \cdot \mu_r^{-1} = 1.172\ldots \tag{5.79}$$

Then the "limit minorant" of the Mahler measure $\mathrm{M}(\beta)$ of β, $\beta > 1$ running over $\mathscr{O}_{\overline{\mathbb{Q}}}$, when $\mathrm{dyg}(\beta)$ tends to infinity, is given by

$$\liminf_{\mathrm{dyg}(\beta)\to\infty} \mathrm{M}(\beta) \geq \Lambda_r \cdot \mu_r = 1.15411\ldots \tag{5.80}$$

Proof. The value $2\arcsin(\kappa(1,a_{\max})/2) = 0.171784\ldots$ is given by Proposition 5.12, and $a_{\max} = 5.8743\ldots$ by Theorem 5.9. The variations of the Mahler measure $\mathrm{M}(\beta)$ of β can be fairly large when β approaches 1^+. On the contrary the lenticular Mahler measure $\mathrm{M}_r(\beta)$ is a continuous function of β on $(1, \theta_{260}^{-1})$ except at the point discontinuities which are the Perron numbers θ_n^{-1} by (5.74), (5.75) and (5.76), writing $n = \mathrm{dyg}(\beta)$ for short.

First, by Proposition 5.12 let us observe that the Riemann-Stieltjes sum

$$S(f,n) := -2\sum_{j=1}^{J_n}\frac{1}{n}\,\mathrm{Log}\left(2\sin\left(\frac{\pi j}{n}\right)\right) = \frac{-1}{\pi}\sum_{j=1}^{J_n}(x_j - x_{j-1})f(x_j)$$

with $x_j = \frac{2\pi j}{n}$ and $f(x) := \mathrm{Log}\left(2\sin\left(\frac{x}{2}\right)\right)$ converges to the limit

$$\lim_{n\to\infty} S(f,n) \;=\; \frac{-1}{\pi}\int_0^{0.171784\ldots} f(x)dx \;=\; \mathrm{Log}\,\Lambda_r \;=\; \mathrm{Log}\,(1.16302\ldots). \qquad (5.81)$$

This limit is a log-sine integral [BBSW] [BS]. Let us now show how Λ_r is related to $\liminf_{\mathrm{dyg}(\beta)\to\infty}\mathrm{M}_r(\beta)$ and $\limsup_{\mathrm{dyg}(\beta)\to\infty}\mathrm{M}_r(\beta)$ to deduce (5.78) and (5.79).

Taking only into account the lenticular zeroes of $P_\beta(z)$, which constitute the lenticulus $\mathscr{L}_\beta$, from Theorem 5.9 and Proposition 5.14, we obtain

$$\mathrm{Log}\,\mathrm{M}_r(\beta) = -\mathrm{Log}\,(\frac{1}{\beta}) - 2\sum_{j=1}^{J_n}\mathrm{Log}\,|\omega_{j,n}| = \mathrm{Log}\,(\frac{1}{\beta}) - 2\sum_{j=1}^{J_n}\mathrm{Log}\,|(\omega_{j,n} - z_{j,n}) + z_{j,n}|$$

$$= \mathrm{Log}\,(\beta) - 2\sum_{j=1}^{J_n}\mathrm{Log}\,|z_{j,n}| - 2\sum_{j=1}^{J_n}\mathrm{Log}\left|1 + \frac{\omega_{j,n} - z_{j,n}}{z_{j,n}}\right|. \qquad (5.82)$$

Obviously the first term of (5.82) tends to 0 when $\mathrm{dyg}(\beta)$ tends to $+\infty$ since $\lim_{n\to\infty}\theta_n = 1$ (Proposition 3.4). Let us turn to the third summation in (5.82). The j-th root $\omega_{j,n} \in \mathscr{L}_\beta$ of $f_\beta(z)$ is the unique root of $f_\beta(z)$ in the disc $D_{j,n} = \{z \mid |\omega_{j,n} - z_{j,n}| < \frac{\pi|z_{j,n}|}{n\, a_{\max}}\}$. From Theorem 5.15 we have the more precise localization in $D_{j,n}$: $|\omega_{j,n} - z_{j,n}| < \frac{\pi|z_{j,n}|}{n\, a_{j,n}}$ for $j = \lceil v_n \rceil, \ldots, J_n$ (main angular sector), with

$$D(\frac{\pi}{a_{j,n}}) = \mathrm{Log}\left[\frac{1 + B_{j,n} - \sqrt{1 - 6B_{j,n} + B_{j,n}^2}}{4B_{j,n}}\right]$$

and $B_{j,n} = 2\sin(\frac{\pi j}{n})\Big(1 - \frac{1}{n}\mathrm{Log}\,(2\sin(\frac{\pi j}{n}))\Big)$ (from (5.42)). For $j = \lceil v_n \rceil, \ldots, J_n$ the following inequalities hold:

$$1 - \frac{1}{n}D(\frac{\pi}{a_{j,n}}) \leq |1 + \frac{\omega_{j,n} - z_{j,n}}{z_{j,n}}| \leq 1 + \frac{1}{n}D(\frac{\pi}{a_{j,n}}),$$

up to second order terms. Let us apply the remainder Theorem of alternating series: for x real, $|x| < 1$, $|\mathrm{Log}\,(1+x) - x| \;\leq\; \frac{x^2}{2}$. Then the third summation in (5.82) satisfies

$$-2\lim_{n\to\infty}\sum_{j=1}^{J_n}\frac{1}{n}\mathrm{Log}\left[\frac{1 + 2\sin(\frac{\pi j}{n}) - \sqrt{1 - 12\sin(\frac{\pi j}{n}) + 4(\sin(\frac{\pi j}{n}))^2}}{8\sin(\frac{\pi j}{n})}\right]$$

$$\leq \liminf_{n\to\infty} \left(-2 \sum_{j=1}^{J_n} \mathrm{Log} \left| 1 + \frac{\omega_{j,n} - z_{j,n}}{z_{j,n}} \right| \right) \tag{5.83}$$

and

$$\limsup_{n\to\infty} \left(-2 \sum_{j=1}^{J_n} \mathrm{Log} \left| 1 + \frac{\omega_{j,n} - z_{j,n}}{z_{j,n}} \right| \right) \leq$$

$$+2 \lim_{n\to\infty} \sum_{j=1}^{J_n} \frac{1}{n} \mathrm{Log} \Big[\frac{1 + 2\sin(\frac{\pi j}{n}) - \sqrt{1 - 12\sin(\frac{\pi j}{n}) + 4(\sin(\frac{\pi j}{n}))^2}}{8\sin(\frac{\pi j}{n})} \Big] \tag{5.84}$$

Let us convert the limits to integrals. The Riemann-Stieltjes sum

$$S(F,n) := -2 \sum_{j=1}^{J_n} \frac{1}{n} \mathrm{Log} \Big[\frac{1 + 2\sin(\frac{\pi j}{n}) - \sqrt{1 - 12\sin(\frac{\pi j}{n}) + 4(\sin(\frac{\pi j}{n}))^2}}{8\sin(\frac{\pi j}{n})} \Big]$$

$$= \frac{-1}{\pi} \sum_{j=1}^{J_n} (x_j - x_{j-1}) F(x_j)$$

with $x_j = \frac{2\pi j}{n}$ and $F(x) := \mathrm{Log} \Big[\frac{1+2\sin(\frac{x}{2}) - \sqrt{1-12\sin(\frac{x}{2})+4(\sin(\frac{x}{2}))^2}}{8\sin(\frac{x}{2})} \Big]$ converges to the limit

$$\lim_{n\to\infty} S(F,n) = \frac{-1}{\pi} \int_0^{0.171784...} F(x)dx = \mathrm{Log}\, \mu_r \quad \text{with} \quad \mu_r = 0.992337\ldots. \tag{5.85}$$

From the inequalities (5.83) and (5.84), with the limit (5.85) as an integral, and by taking the exponential of (5.82), we obtain the two multiplicative factors μ_r and μ_r^{-1} of Λ_r in (5.78), resp. in (5.79).

Let us show that the second summation in (5.82) gives the term Λ_r in the inequalities (5.78) and (5.79), when n tends to infinity. From (5.81) it will suffice to show that

$$\lim_{n\to\infty} S(f,n) = -2 \lim_{n\to\infty} \sum_{j=1}^{J_n} \mathrm{Log} |z_{j,n}| \tag{5.86}$$

The identity (5.86) only concerns the roots of the trinomials G_n. It was already proved to be true, but with $\lfloor n/6 \rfloor$ instead of J_n as maximal index j, in the summation, in [VG6] §4.2, pp 111–115. The arguments of the proof are the same, the domain of integration being now $(0, \lim_{n\to\infty} 2\pi \frac{J_n}{n}]$ given by Proposition 5.12. $\square$

5.5 Poincaré asymptotic expansion of the lenticular Mahler measure

The aim of this subsection is to prove Theorem 5.27, in the continuation of the last paragraph.

The logarithm of the lenticular Mahler measure $\mathrm{M}_r(\beta)$ of $\beta > 1$, with $\mathrm{dyg}(\beta) \geq 260$, given by (5.82), admits the lower bound

$$L_r(\beta) = \mathrm{Log}\,(\beta) - 2\sum_{j=1}^{J_n} \mathrm{Log}\,|z_{j,n}| - 2\sum_{j=1}^{\lfloor v_n \rfloor} \mathrm{Log}\,(1+\frac{\pi}{n\,a_{max}}) - 2\sum_{j=\lceil v_n \rceil}^{J_n} \mathrm{Log}\,(1+\frac{\pi}{n\,a_{j,n}}) \tag{5.87}$$

which is only a function of $n = \mathrm{dyg}(\beta)$, where $(a_{j,n})$ is given by Theorem 5.15, the sequence (v_n) by the Appendix, and J_n by Definition 5.11 and Proposition 5.12. From (5.83), (5.85) and (5.86), the limit is $\lim_{\mathrm{dyg}(\beta)\to\infty} L_r(\beta) = \mathrm{Log}\,\Lambda_r + \mathrm{Log}\,\mu_r$. In Theorem 5.27, we will gather the asymptotic contributions of each term and obtain the asymptotic expansion of $L_r(\beta)$ as a function of n.

(i) First term in (5.87): from Lemma 3.5 and Theorem 5.2,

$$\mathrm{Log}\,(\beta) = \frac{\mathrm{Log}\,n}{n}(1-\lambda_n) + \frac{1}{n} O\left(\left(\frac{\mathrm{Log}\,\mathrm{Log}\,n}{\mathrm{Log}\,n}\right)^2\right) = O\left(\frac{\mathrm{Log}\,n}{n}\right); \tag{5.88}$$

(ii) second term in (5.87): from Proposition 3.8, $\sum_{j=\lceil v_n \rceil}^{J_n} \mathrm{Log}\,|z_{j,n}| =$

$$\sum_{j=\lceil v_n \rceil}^{J_n} \mathrm{Log}\left(1 + \frac{1}{n}\mathrm{Log}\left(2\sin(\frac{\pi j}{n})\right) + \frac{1}{2n}\left(\frac{\mathrm{Log}\,\mathrm{Log}\,n}{\mathrm{Log}\,n}\right)^2 + \frac{1}{n}O\left(\frac{(\mathrm{Log}\,\mathrm{Log}\,n)^2}{(\mathrm{Log}\,n)^3}\right)\right)$$

with the constant 1 involved in the Big O. Let us apply the remainder Theorem of alternating series: for x real, $|x| < 1$, $|\mathrm{Log}\,(1+x) - x| \leq \frac{x^2}{2}$. Then

$$\left| \sum_{j=\lceil v_n \rceil}^{J_n} \mathrm{Log}\,|z_{j,n}| - \sum_{j=\lceil v_n \rceil}^{J_n} \frac{1}{n}\mathrm{Log}\left(2\sin(\frac{\pi j}{n})\right) - \sum_{j=\lceil v_n \rceil}^{J_n} \frac{1}{2n}\left(\frac{\mathrm{Log}\,\mathrm{Log}\,n}{\mathrm{Log}\,n}\right)^2 \right|$$

$$\leq \sum_{j=\lceil v_n \rceil}^{J_n} \frac{1}{n}\left| O\left(\frac{(\mathrm{Log}\,\mathrm{Log}\,n)^2}{(\mathrm{Log}\,n)^3}\right)\right|$$

$$+\frac{1}{2}\sum_{j=\lceil v_n \rceil}^{J_n} \frac{1}{n^2}\left[\mathrm{Log}\left(2\sin(\frac{\pi j}{n})\right) + \frac{1}{2}\left(\frac{\mathrm{Log}\,\mathrm{Log}\,n}{\mathrm{Log}\,n}\right)^2 + O\left(\frac{(\mathrm{Log}\,\mathrm{Log}\,n)^2}{(\mathrm{Log}\,n)^3}\right)\right]^2. \tag{5.89}$$

For $1 \leq j \leq J_n$, the inequalities $0 < 2\sin(\pi j/n) \leq 1$ and $\mathrm{Log}\,(2\sin(\pi j/n)) < 0$ hold. Then $|\mathrm{Log}\,(2\sin(\pi j/n)| \leq |\mathrm{Log}\,(2\sin(\pi/n))| = O(\mathrm{Log}\,n)$. On the other hand, the two $O(\ )$s in the rhs of (5.89) involve a constant which does not depend upon j. Therefore, from Proposition 5.12, the rhs of (5.89) is

$$= O\left(\left(\frac{(\mathrm{Log}\,\mathrm{Log}\,n)^2}{(\mathrm{Log}\,n)^3}\right)\right) + O\left(\frac{\mathrm{Log}^2 n}{n}\right) = O\left(\left(\frac{(\mathrm{Log}\,\mathrm{Log}\,n)^2}{(\mathrm{Log}\,n)^3}\right)\right).$$

On the other hand, the two regimes of asymptotic expansions in the Bump give (Appendix)

$$\sum_{j=\lceil u_n \rceil}^{\lfloor v_n \rfloor} \mathrm{Log}\,|z_{j,n}| = O\left(\frac{(\mathrm{Log}\,n)^{2+\varepsilon}}{n}\right), \quad \sum_{j=1}^{\lfloor u_n \rfloor} \mathrm{Log}\,|z_{j,n}| = O\left(\frac{(\mathrm{Log}\,n)^{2}}{n}\right)$$

and

$$\sum_{j=\lceil \mathrm{Log}\,n \rceil}^{\lceil v_n \rceil} \frac{2}{n} \mathrm{Log}\left(2\sin(\frac{\pi j}{n})\right) = O\left(\frac{(\mathrm{Log}\,n)^{2+\varepsilon}}{n}\right).$$

Therefore

$$-2\sum_{j=1}^{J_n} \mathrm{Log}\,|z_{j,n}| = -\sum_{j=\lceil \mathrm{Log}\,n \rceil}^{J_n} \frac{2}{n} \mathrm{Log}\left(2\,\sin(\frac{\pi j}{n})\right) + O\left(\left(\frac{\mathrm{Log}\,\mathrm{Log}\,n}{\mathrm{Log}\,n}\right)^2\right) \tag{5.90}$$

with the constant $\frac{1}{2\pi}\arcsin(\frac{\kappa}{2})$ (from Proposition 5.12) involved in the Big O.
(iii) third term in (5.87): with the definition of ε and (v_n) (Appendix),

$$-2\sum_{j=1}^{\lfloor v_n \rfloor} \mathrm{Log}\,(1+\frac{\pi}{n\,a_{max}}) = O\left(\frac{(\mathrm{Log}\,n)^{1+\varepsilon}}{n}\right); \tag{5.91}$$

(iv) fourth term in (5.87): from the Theorem of alternating series,

$$\left|\sum_{j=\lceil v_n \rceil}^{J_n} \mathrm{Log}\,(1+\frac{\pi}{n\,a_{j,n}}) - \sum_{j=\lceil v_n \rceil}^{J_n} \frac{1}{n} D(\frac{\pi}{a_{j,n}}) - \sum_{j=\lceil v_n \rceil}^{J_n} \frac{1}{n} \mathrm{tl}(\frac{\pi}{a_{j,n}})\right| \leq \frac{1}{2}\sum_{j=\lceil v_n \rceil}^{J_n} \left(\frac{\pi}{n\,a_{j,n}}\right)^2. \tag{5.92}$$

The terminant $\mathrm{tl}(\frac{\pi}{a_{j,n}}) = O\left(\frac{(\mathrm{Log}\,\mathrm{Log}\,n)^2}{(\mathrm{Log}\,n)^3}\right)$ is given by (5.44). From Theorem 5.15, with $B_{j,n} = 2\sin(\frac{\pi j}{n})\left(1 - \frac{1}{n}\mathrm{Log}\,(2\sin(\frac{\pi j}{n}))\right)$, it is easy to show

$$D(\frac{\pi}{a_{j,n}}) = \mathrm{Log}\left[\frac{1+B_{j,n} - \sqrt{1-6B_{j,n}+B_{j,n}^2}}{4B_{j,n}}\right]$$

$$= \mathrm{Log}\left[\frac{1+2\sin(\frac{\pi j}{n}) - \sqrt{1-12\sin(\frac{\pi j}{n})+4\sin(\frac{\pi j}{n})^2}}{8\sin(\frac{\pi j}{n})}\right] + O\left(\frac{\mathrm{Log}\,n}{n}\right).$$

The rhs of (5.92) is $= O\left(\frac{1}{n}\right)$. Then $-2\sum_{j=\lceil v_n \rceil}^{J_n} \mathrm{Log}\,(1+\frac{\pi}{n\,a_{j,n}}) =$

$$\sum_{j=\lceil v_n \rceil}^{J_n} \frac{-2}{n} \mathrm{Log}\left[\frac{1+2\sin(\frac{\pi j}{n}) - \sqrt{1-12\sin(\frac{\pi j}{n})+4\sin(\frac{\pi j}{n})^2}}{8\sin(\frac{\pi j}{n})}\right] + O\left(\frac{(\mathrm{Log}\,\mathrm{Log}\,n)^2}{(\mathrm{Log}\,n)^3}\right). \tag{5.93}$$

The summation $\sum_{j=\lceil v_n \rceil}^{J_n}$ can be replaced by $\sum_{j=\lceil \mathrm{Log}\, n \rceil}^{J_n}$. Indeed, from the definition of the sequence (v_n) (Appendix),

$$\sum_{j=\lceil \mathrm{Log}\, n \rceil}^{\lceil v_n \rceil} \frac{2}{n} \mathrm{Log}\Big[\frac{1+2\sin(\frac{\pi j}{n}) - \sqrt{1-12\sin(\frac{\pi j}{n})+4\sin(\frac{\pi j}{n})^2}}{8\sin(\frac{\pi j}{n})}\Big] = O\Big(\frac{(\mathrm{Log}\, n)^{2+\varepsilon}}{n}\Big).$$

Inserting the contributions (5.88) (5.90) (5.91) (5.93) in (5.87) leads to

$$L_r(\beta) = \mathrm{Log}\,\Lambda_r + \mathrm{Log}\,\mu_r + \Big(-\mathrm{Log}\,\Lambda_r - \sum_{j=\lceil \mathrm{Log}\, n \rceil}^{J_n} \frac{2}{n} \mathrm{Log}\,\big(2\,\sin(\frac{\pi j}{n})\big)\Big)$$

$$+\Big(-\mathrm{Log}\,\mu_r - \sum_{j=\lceil \mathrm{Log}\, n \rceil}^{J_n} \frac{2}{n} \mathrm{Log}\,\big(\frac{1+2\sin(\frac{\pi j}{n}) - \sqrt{1-12\sin(\frac{\pi j}{n})+4\sin(\frac{\pi j}{n})^2}}{8\sin(\frac{\pi j}{n})}\big)\Big)$$

$$+O\Big(\Big(\frac{\mathrm{Log}\,\mathrm{Log}\, n}{\mathrm{Log}\, n}\Big)^2\Big) \tag{5.94}$$

with the constant $\frac{1}{2\pi}\arcsin(\frac{\kappa}{2})$ involved in the Big O. Let us denote by Δ_1 the first term within brackets, resp. Δ_2 the second term within brackets, in (5.94) so that

$$D(L_r(\beta)) = \mathrm{Log}\,(\Lambda_r \mu_r) + \Delta_1 + \Delta_2. \tag{5.95}$$

Calculation of $|\Delta_1|$*:* let us estimate and give an upper bound of $|\Delta_1| =$

$$\Big|\frac{-1}{\pi}\int_0^{2\arcsin(\kappa/2)} \mathrm{Log}\,\Big(2\sin(x/2)\Big)dx - \sum_{j=\lceil \mathrm{Log}\, n \rceil}^{J_n} \frac{-2}{n} \mathrm{Log}\,\big(2\,\sin(\frac{\pi j}{n})\big)\Big|. \tag{5.96}$$

In (5.96) the sums are truncated Riemann-Stieltjes sums of $\mathrm{Log}\,\Lambda_r$, the integral being $\mathrm{Log}\,\Lambda_r$. Referring to Stoer and Bulirsch ([SB], pp 126–128) we now replace $\mathrm{Log}\,\Lambda_r$ by an approximate value obtained by integration of an interpolation polynomial by the methods of Newton-Cotes; we just need to know this approximate value up to $O\big(\big(\frac{\mathrm{Log}\,\mathrm{Log}\, n}{\mathrm{Log}\, n}\big)^2\big)$. Up to $O\big(\big(\frac{\mathrm{Log}\,\mathrm{Log}\, n}{\mathrm{Log}\, n}\big)^2\big)$, we will show that:

(i–1) *an upper bound of* (5.96) *is (*κ *stands for* $\kappa(1, a_{max})$ *as in* Proposition 5.12)

$$\frac{\arcsin(\kappa/2)}{\pi}\,\frac{1}{\mathrm{Log}\, n},$$

(ii–1) *the approximate value of* $\mathrm{Log}\,\Lambda_r$ *is independent of the integer* m *(i.e. step length) used in the Newton-Cotes formulas, assuming the weights* $(\alpha_q)_{q=0,1,\ldots,m}$ *associated with* m *all positive. Indeed, if* m *is arbitrarily large, the estimate of the integral should be very good by these methods, ideally exact at the limit (*m"$=$"$+\infty$*).*

Proof of (i–1): we consider the decomposition of the interval of integration as

$\big(0, 2\arcsin(\kappa/2)\big] =$

$$\Big(0, \frac{2\pi\lceil \mathrm{Log}\, n\rceil}{n}\Big] \cup \Big(\bigcup_{j=\lceil \mathrm{Log}\, n\rceil}^{J_n-1} \Big[\frac{2\pi j}{n}, \frac{2\pi(j+1)}{n}\Big]\Big) \cup \Big[\frac{2\pi J_n}{n}, 2\arcsin(\kappa/2)\Big] \tag{5.97}$$

and proceed by calcutating the estimations of

$$\left| \frac{-1}{\pi} \int_{\frac{2\pi j}{n}}^{\frac{2\pi(j+1)}{n}} \mathrm{Log}\Big(2\sin(x/2)\Big)dx - \frac{-2}{n}\,\mathrm{Log}\big(2\,\sin(\frac{\pi j}{n})\big)\right| \tag{5.98}$$

on the intervals $\mathscr{I}_j := \big[\frac{2\pi j}{n}, \frac{2\pi(j+1)}{n}\big]$, $j = \lceil \mathrm{Log}\, n\rceil, \lceil \mathrm{Log}\, n\rceil + 1, \ldots, J_n - 1$. On each such $\mathscr{I}_j$, the function $f(x)$ is approximated by its interpolation polynomial $P_m(x)$, where $m \geq 1$ is the number of subintervals forming an uniform partition of $\mathscr{I}_j$ given by

$$y_q = \frac{2\pi j}{n} + q\frac{2\pi}{n}\frac{1}{m}, \qquad q = 0, 1, \ldots, m, \tag{5.99}$$

of step length $h_{NC} := \frac{2\pi}{nm}$, and P_m the interpolating polynomial of degree m or less with

$$P_m(y_q) = f(y_q), \qquad \text{for } q = 0, 1, \ldots, m.$$

The Newton-Cotes formulas

$$\int_{\frac{2\pi j}{n}}^{\frac{2\pi(j+1)}{n}} P_m(x)dx = h_{NC}\sum_{q=0}^{m} \alpha_q f(y_q)$$

provide approximate values of $\int_{\frac{2\pi j}{n}}^{\frac{2\pi(j+1)}{n}} f(x)dx$, where the α_q are the weights obtained by integrating the Lagrange's interpolation polynomials. Steffensen [Sff] ([SB], p 127) showed that the approximation error may be expressed as follows:

$$\int_{\frac{2\pi j}{n}}^{\frac{2\pi(j+1)}{n}} P_m(x)dx - \int_{\frac{2\pi j}{n}}^{\frac{2\pi(j+1)}{n}} f(x)dx = h_{NC}^{p+1}\cdot K \cdot f^{(p+1)}(\xi), \qquad \xi \in \overset{o}{\mathscr{I}_j},$$

where $p \geq 2$ is an integer related to m, and K a constant.

Using [SB], p. 128, and $m = 1$, the method being the "Trapezoidal rule", we have: "$p = 2, K = 1/12, \alpha_0 = \alpha_1 = 1/2$". Then (5.98) is estimated by

$$\left|\frac{1}{2}\frac{2\pi}{n}\left[\frac{-1}{\pi}\,\mathrm{Log}\big(2\,\sin(\frac{\pi j}{n})\big) + \frac{-1}{\pi}\,\mathrm{Log}\big(2\,\sin(\frac{\pi(j+1)}{n})\big)\right] - \frac{-2}{n}\,\mathrm{Log}\big(2\,\sin(\frac{\pi j}{n})\big)\right|$$

$$= \frac{1}{n}\left|\mathrm{Log}\big(2\,\sin(\frac{\pi j}{n})\big) - \mathrm{Log}\big(2\,\sin(\frac{\pi(j+1)}{n})\big)\big)\right| = \frac{2\pi}{n^2}\left|\frac{\cos(\xi/2)}{2\sin(\xi/2)}\right| \leq \frac{1}{n}\frac{1}{\mathrm{Log}\, n} \tag{5.100}$$

for some $\xi \in \overset{o}{\mathscr{I}_j}$, for large n. The (Steffensen's) approximation error "$h_{NC}^3 \cdot (1/12) \cdot f^{(2)}(\xi)$" for the trapezoidal rule, relative to (5.98), is

$$\frac{1}{\pi}\left(\frac{2\pi}{n}\right)^3 \frac{1}{12}\left|\frac{-1}{4\sin^2(\xi/2)}\right| \leq \frac{1}{6n}\frac{1}{(\mathrm{Log}\, n)^2}. \tag{5.101}$$

By Proposition 5.12 the integral

$$\left|\frac{-1}{\pi}\int_{\frac{2\pi J_n}{n}}^{2\arcsin\kappa/2} \mathrm{Log}\left(2\sin(x/2)\right)dx\right| \qquad \text{is a} \qquad O\left(\frac{1}{n}\right).$$

Then, summing up the contributions of all the intervals $\mathscr{I}_j$, we obtain the following upper bound of (5.96)

$$\left|\frac{-1}{\pi}\int_0^{(2\pi\mathrm{Log}\, n)/n} \mathrm{Log}\left(2\sin(x/2)\right)dx\right| + \frac{\arcsin(\kappa/2)}{\pi}\frac{1}{\mathrm{Log}\, n}. \tag{5.102}$$

with global (Steffensen's) approximation error, from (5.101),

$$O\left(\frac{1}{(\mathrm{Log}\, n)^2}\right)$$

By integrating by parts the integral in (5.102), for large n, it is easy to show that this integral is $= O\left(\frac{(\mathrm{Log}\, n)^2}{n}\right)$. We deduce the following asymptotic expansion

$$\Delta_1 = \frac{\mathscr{R}}{\mathrm{Log}\, n} + O\left(\frac{1}{(\mathrm{Log}\, n)^2}\right) \qquad \text{with} \quad |\mathscr{R}| < \frac{\arcsin(\kappa/2)}{\pi}. \tag{5.103}$$

Proof of (ii–1): Let us show that the upper bound $\frac{\arcsin(\kappa/2)}{\pi}\frac{1}{\mathrm{Log}\, n}$ is independent of the integer m used, once assumed the positivity of the weights $(\alpha_q)_{q=0,1,\ldots,m}$. For $m \geq 1$ fixed, this is merely a consequence of the relation between the weights in the Newton-Cotes formulas. Indeed, we have $\sum_{q=0}^m \alpha_q = m$, and therefore

$$\left|\int_{\frac{2\pi j}{n}}^{\frac{2\pi(j+1)}{n}} P_m(x)dx - h_{NC}m f(y_0)\right| = h_{NC}\left|\sum_{q=0}^m \alpha_q(f(y_q) - f(y_0))\right|$$

$$\leq h_{NC}\left(\sum_{q=0}^m |\alpha_q|\right) \sup_{\xi\in\mathscr{L}_j}\left|f'(\xi)\right|.$$

Since $h_{NC}m = \frac{2\pi}{n}$ and that the inequality $\sup_{\xi\in\mathscr{L}_j}|f'(\xi)| \leq |f'((2\pi\mathrm{Log}\, n)/n)|$ holds uniformly for all j, we deduce the same upper bound as in (5.100) for the Trapezoidal rule. Summing up the contributions over all the intervals $\mathscr{I}_j$, we obtain the same upper bound (5.102) of (5.96) as before.

As for the (Steffensen's) approximation errors, they make use of the successive derivatives of the function $f(x) = \mathrm{Log}\,(2\sin(x/2))$. We have:

$$f'(x) = \frac{\cos(x/2)}{2\sin(x/2)},\ \ f''(x) = -\frac{1}{4\sin^2(x/2)},\ \ f'''(x) = \frac{\cos(x/2)}{4\sin^3(x/2)}\cdots$$

Recursively, it is easy to show that the q-th derivative of $f(x)$, $q \geq 1$, is a rational function of the two quantities $\cos(x/2)$ and $\sin(x/2)$ with bounded numerator on the interval $(0, \pi/3]$, and a denominator which is $\sin^q(x/2)$. For the needs of majoration in the Newton-Cotes formulas over each interval of the collection $(\mathscr{I}_j)$, this denominator takes its smallest value at $\xi = (2\pi\lceil \mathrm{Log}\,n\rceil)/n$. Therefore, for large n, the (Steffensen's) approximation error "$h_{NC}^{p+1}\cdot K\cdot f^{(p)}(\xi)$" on one interval $\mathscr{I}_j$ is

$$O\Big(\Big(\frac{2\pi}{nm}\Big)^{p+1}\cdot K\cdot\frac{n^p}{(\pi\,\mathrm{Log}\,n)^p}\Big) = O\Big(\frac{1}{n(\mathrm{Log}\,n)^p}\Big).$$

By summing up over the intervals $\mathscr{I}_j$, we obtain the global (Steffensen's) approximation error ($p \geq 2$)

$$O\Big(\frac{1}{(\mathrm{Log}\,n)^p}\Big) \qquad \text{which is a} \qquad O\Big(\Big(\frac{\mathrm{Log}\,\mathrm{Log}\,n}{\mathrm{Log}\,n}\Big)^2\Big).$$

Calculation of $|\Delta_2|$: we proceed as above for establishing an upper bound of

$$|\Delta_2| = \Big|\frac{-1}{\pi}\int_0^{2\arcsin(\frac{\kappa(1,a_{\max})}{2})}\mathrm{Log}\Big[\frac{1+2\sin(\frac{x}{2})-\sqrt{1-12\sin(\frac{x}{2})+4(\sin(\frac{x}{2}))^2}}{8\sin(\frac{x}{2})}\Big]dx$$

$$-\sum_{j=\lceil \mathrm{Log}\,n\rceil}^{J_n}\frac{-2}{n}\,\mathrm{Log}\,\Big(\frac{1+2\sin(\frac{\pi j}{n})-\sqrt{1-12\sin(\frac{\pi j}{n})+4\sin(\frac{\pi j}{n})^2}}{8\sin(\frac{\pi j}{n})}\Big)\Big| \qquad (5.104)$$

In (5.104) the sums are truncated Riemann-Stieltjes sums of $\mathrm{Log}\,\mu_r$, the integral being $\mathrm{Log}\,\mu_r$. As above, the methods of Newton-Cotes (Stoer and Bulirsch ([SB], pp 126–128) will be applied to compute an approximate value of the integral up to $O\big(\big(\frac{\mathrm{Log}\mathrm{Log}\,n}{\mathrm{Log}\,n}\big)^2\big)$. Up to $O\big(\big(\frac{\mathrm{Log}\mathrm{Log}\,n}{\mathrm{Log}\,n}\big)^2\big)$, we will show that:

(i–2) *an upper bound of* (5.104) *is (*κ *stands for* $\kappa(1, a_{max})$ *as in* Proposition 5.12)

$$\frac{4\,\arcsin(\kappa/2)}{\kappa\sqrt{2\kappa(3-\kappa)\mathrm{Log}\,(1/\kappa)}}\,\frac{1}{\sqrt{n}} \qquad \textit{which is a}\ \ O\Big(\Big(\frac{\mathrm{Log}\,\mathrm{Log}\,n}{\mathrm{Log}\,n}\Big)^2\Big), \qquad (5.105)$$

in other terms that (5.104) *is equal to zero up to* $O\big(\big(\frac{\mathrm{Log}\mathrm{Log}\,n}{\mathrm{Log}\,n}\big)^2\big)$,

(ii–2) *the approximate value of* $\mathrm{Log}\,\mu_r$ *is independent of the step length* m *used in the Newton-Cotes formulas, assuming the weights* $(\alpha_q)_{q=0,1,\ldots,m}$ *associated with* m *all positive*.

Proof of (i–2): The decomposition of the interval of integration $\big(0, 2\arcsin(\kappa/2)\big]$ remains the same as above, given by (5.97). Let us treat the complete interval of integration $\big(0, 2\arcsin(\kappa/2)\big]$ by subintervals. We first proceed by estimating an upper bound of

$$\Big|\frac{-1}{\pi}\int_{\frac{2\pi j}{n}}^{\frac{2\pi(j+1)}{n}} \mathrm{Log}\Big[\frac{1+2\sin(\frac{x}{2})-\sqrt{1-12\sin(\frac{x}{2})+4(\sin(\frac{x}{2}))^2}}{8\sin(\frac{x}{2})}\Big]dx$$

$$-\frac{-2}{n}\,\mathrm{Log}\Big(\frac{1+2\sin(\frac{\pi j}{n})-\sqrt{1-12\sin(\frac{\pi j}{n})+4\sin(\frac{\pi j}{n})^2}}{8\sin(\frac{\pi j}{n})}\Big)\Big| \tag{5.106}$$

on the intervals $\mathscr{I}_j := \big[\frac{2\pi j}{n}, \frac{2\pi(j+1)}{n}\big]$, $j = \lceil \mathrm{Log}\, n\rceil, \lceil \mathrm{Log}\, n\rceil + 1, \ldots, J_n - 1$. Let

$$F(x) := \mathrm{Log}\Big[\frac{1+2\sin(\frac{x}{2})-\sqrt{1-12\sin(\frac{x}{2})+4(\sin(\frac{x}{2}))^2}}{8\sin(\frac{x}{2})}\Big].$$

On each interval $\mathscr{I}_j$ the function $F(x)$ is approximated by its interpolation polynomial (say) $P_{F,m}(x)$, where $m \geq 1$ is the number of subintervals of $\mathscr{I}_j$ given by their extremities y_q by (5.99), of step length $h_{NC} := \frac{2\pi}{nm}$, and $P_{F,m}$ the interpolating polynomial of degree m or less with

$$P_{F,m}(y_q) = F(y_q), \qquad \text{for } q = 0, 1, \ldots, m.$$

The Newton-Cotes formulas

$$\int_{\frac{2\pi j}{n}}^{\frac{2\pi(j+1)}{n}} P_{F,m}(x)dx = h_{NC}\sum_{q=0}^{m}\alpha_q F(y_q) \tag{5.107}$$

provide the approximate values $\int_{\frac{2\pi j}{n}}^{\frac{2\pi(j+1)}{n}} F(x)dx$, where the α_qs are the weights obtained by integrating the Lagrange's interpolation polynomials. Using [SB], p. 128, and $m = 1$, the method being the "Trapezoidal rule", we have: $p = 2$, $K = 1/12$, $\alpha_0 = \alpha_1 = 1/2$. Then (5.106) is estimated by

$$\Big|\frac{1}{2}\frac{2\pi}{n}\Big[\frac{-1}{\pi}F\big(\frac{2\pi j}{n}\big)+\frac{-1}{\pi}F\big(\frac{2\pi(j+1)}{n}\big)\Big]-\frac{-2}{n}F\big(\frac{2\pi j}{n}\big)\Big|$$

$$= \frac{1}{n}\Big|F\big(\frac{2\pi j}{n}\big)-F\big(\frac{2\pi(j+1)}{n}\big)\big)\Big| = \frac{2\pi}{n^2}\left|F'(\xi)\right| \tag{5.108}$$

for some $\xi \in \overset{o}{\mathscr{I}_j}$, for large n. As in Remark 5.13, let $x = 2\arcsin(\kappa/2)$. The derivative

$$F'(y) = \frac{\cos(y/2)(-2\sin(y/2)+1-\sqrt{4\sin^2(y/2)-12\sin(y/2)+1}}{4\sin(y/2)\sqrt{4\sin^2(y/2)-12\sin(y/2)+1)}} > 0 \tag{5.109}$$

is increasing on the interval $(0,x)$. When $y = \frac{2\pi J_n}{n} < x$ tends to x^-, by Proposition 5.12 and Remark 5.13, since $0 < \sqrt{4\sin^2(y/2) - 12\sin(y/2) + 1} \le 1$ is close to zero for $y = 2\pi J_n/n$, the following inequality holds

$$F'(\frac{2\pi J_n}{n}) \le \frac{2/\kappa}{\sqrt{4\sin^2(\frac{\pi J_n}{n}) - 12\sin(\frac{\pi J_n}{n}) + 1}}. \tag{5.110}$$

The upper bound is a function of n which comes from the asymptotic expansion of $\frac{\pi J_n}{n} - \frac{x}{2}$, as deduced from (5.32). Indeed, from (5.32) and using Remark 5.13 (ii),

$$4\sin^2(\frac{\pi J_n}{n}) - 12\sin(\frac{\pi J_n}{n}) + 1 = (\frac{\pi J_n}{n} - \frac{x}{2})[8\sin(x/2)\cos(x/2) - 12\cos(x/2)] + O(\frac{1}{n^2})$$

$$= \frac{2\kappa(3-\kappa)\mathrm{Log}\,(1/\kappa)}{n} + \frac{1}{n} O((\frac{\mathrm{Log}\,\mathrm{Log}\, n}{\mathrm{Log}\, n})^2) \tag{5.111}$$

From (5.110) and (5.111) we deduce $|F'(\frac{2\pi J_n}{n})| < \frac{(2/\kappa)}{\sqrt{2\kappa(3-\kappa)\mathrm{Log}\,(1/\kappa)}} \sqrt{n}$. From (5.108), we deduce the following upper bound of (5.106) on each $\mathscr{I}_j := \left[\frac{2\pi j}{n}, \frac{2\pi(j+1)}{n}\right]$:

$$\frac{4\pi}{\kappa\sqrt{2\kappa(3-\kappa)\mathrm{Log}\,(1/\kappa)}} \frac{1}{n^{3/2}}. \tag{5.112}$$

By summing up the contributions, for $j = \lceil \mathrm{Log}\, n \rceil, \ldots, J_n - 1$, from (5.112) and the asymptotics of J_n given by (5.32), we deduce the upper bound (5.105) of $|\Delta_2|$.

Let us prove that the method of numerical integration we use leads to a (Steffensen's) approximation error which is a $O\big((\frac{\mathrm{Log}\,\mathrm{Log}\, n}{\mathrm{Log}\, n})^2\big)$. The (Steffensen's) approximation error "$h_{NC}^3 \cdot (1/12) \cdot F^{(2)}(\xi)$" for the trapezoidal rule applied to (5.106) ([SB], p. 127–128) is

$$\frac{1}{\pi}\left(\frac{2\pi}{n}\right)^3 \frac{1}{12}\left|F^{(2)}(\xi)\right| \qquad \text{for some } \xi \in \overset{o}{\mathscr{I}_j}. \tag{5.113}$$

The second derivative $F''(y)$ is positive and increasing on $(0, \frac{2\pi J_n}{n})$. It is easy to show that there exists a constant $C > 0$ such that

$$F''(\frac{2\pi J_n}{n}) \le \frac{C}{(4\sin^2(\frac{\pi J_n}{n}) - 12\sin(\frac{\pi J_n}{n}) + 1)^{3/2}}.$$

Using the asymptotic expansion of J_n ((5.32); Remark 5.13 (ii); (5.111)), there exist $C_1 > 0$ such that

$$F''(\frac{2\pi J_n}{n}) \le C_1\, n^{3/2}. \tag{5.114}$$

From (5.113) and (5.114), summing up the contributions for $j = \lceil \mathrm{Log}\, n \rceil, \ldots, J_n - 1$, the global (Steffensen's) approximation error of (5.104) for $|\Delta_2|$ admits the following

upper bound, for some constants $C_2' > 0, C_2 > 0$,

$$C_2' \frac{J_n}{n^3} n^{3/2} = C_2 \frac{1}{\sqrt{n}} \qquad \text{which is a } O\big(\big(\frac{\operatorname{Log}\operatorname{Log} n}{\operatorname{Log} n}\big)^2\big).$$

Now let us turn to the extremity intervals. Using the Appendix, and (5.32) in Proposition 5.12, it is easy to show that the two integrals

$$\frac{-1}{\pi}\int_0^{\frac{2\pi\lceil \operatorname{Log} n\rceil}{n}} \quad \text{and} \quad \frac{-1}{\pi}\int_{\frac{2\pi J_n}{n}}^{2\arcsin(\kappa/2)} \qquad \text{are} \qquad O\left(\left(\frac{\operatorname{Log}\operatorname{Log} n}{\operatorname{Log} n}\right)^2\right).$$

Proof of (ii–2): On each interval $\mathscr{I}_j := \big[\frac{2\pi j}{n}, \frac{2\pi(j+1)}{n}\big]$, $j = \lceil \operatorname{Log} n\rceil, \ldots, J_n - 1$, let us assume that the number m of subintervals of $\mathscr{I}_j$ given by their extremities y_q by (5.99), is ≥ 2. The weights α_q in (5.107) are assumed to be positive.

The upper bound $\frac{4\arcsin(\kappa/2)}{\kappa\sqrt{2\kappa(3-\kappa)\operatorname{Log}(1/\kappa)}} \frac{1}{\sqrt{n}}$ of (5.104) is independent of $m \geq 2$, once assumed the positivity of the weights $(\alpha_q)_{q=0,1,\ldots,m}$, since, due to the relation between the weights in the Newton-Cotes formulas $\sum_{q=0}^m \alpha_q = m$,

$$\left|\int_{\frac{2\pi j}{n}}^{\frac{2\pi(j+1)}{n}} P_m(x)dx - h_{NC} m F(y_0)\right| = h_{NC}\left|\sum_{q=0}^{m} \alpha_q (F(y_q) - F(y_0))\right|$$

$$\leq h_{NC}\big(\sum_{q=0}^{m} |\alpha_q|\big) \sup_{\xi\in\mathscr{L}_j} \big|F'(\xi)\big|.$$

Since $h_{NC} m = \frac{2\pi}{n}$ and that $\sup_{\xi\in\mathscr{L}_j}|F'(\xi)| \leq |F'((2\pi J_n)/n)|$ holds uniformly for all $j = \lceil \operatorname{Log} n\rceil, \ldots, J_n - 1$, we deduce the same upper bound (5.112) as for the Trapezoidal rule. Summing up the contributions over all the intervals $\mathscr{I}_j$, we obtain the same upper bound (5.105) of (5.104), as before.

As for the (Steffensen's) approximation errors involved in the numerical integration (5.107) there are "$h_{NC}^{p+1} \cdot K \cdot F^{(p)}(\xi)$" on one interval $\mathscr{I}_j$, for some $p \geq 2$. They make use of the successive derivatives of the function $F(x)$. It can be shown that they contribute negligibly, after summing up over all the intervals $\mathscr{I}_j$, as $O\left(\left(\frac{\operatorname{Log}\operatorname{Log} n}{\operatorname{Log} n}\right)^2\right)$.

Gathering the different terms from (i–1)(i–2), the Steffenssen's error terms and the error terms due to the numerical integration by the Newton-Cotes method (ii–1)(ii–2), we have proved the following theorem.

Theorem 5.27. *Let $\beta > 1$ be an algebraic integer such that $n = \operatorname{dyg}(\beta) \geq 260$. The asymptotic expansion of the minorant $L_r(\beta)$ of $\operatorname{Log}\mathrm{M}_r(\beta)$ is*

$$L_r(\beta) = \operatorname{Log}\Lambda_r\mu_r + \frac{\mathscr{R}}{\operatorname{Log} n} + O\big(\big(\frac{\operatorname{Log}\operatorname{Log} n}{\operatorname{Log} n}\big)^2\big), \quad \textit{with} \quad |\mathscr{R}| < \frac{\arcsin(\kappa/2)}{\pi} \tag{5.115}$$

and $\mathscr{R}$ depending upon β and n.

5.6 A Dobrowolski type minoration

The first two terms of $L_r(\beta)$ in (5.115) provide the following Dobrowolski type minoration of the Mahler measure $\mathrm{M}(\beta) \geq \mathrm{M}_r(\beta)$.

Theorem 5.28. *Let* $\beta > 1$ *be an algebraic integer such that* $\mathrm{dyg}(\beta) \geq 260$. *Then*

$$\mathrm{M}(\beta) \geq \Lambda_r \mu_r - \frac{\Lambda_r \mu_r \arcsin(\kappa/2)}{\pi} \, \frac{1}{\mathrm{Log}\,(\mathrm{dyg}(\beta))} \tag{5.116}$$

Proof. Taking the exponential of (5.115) gives

$$\mathrm{M}_r(\beta) \geq \exp(L_r(\beta)) = \Lambda_r \mu_r \Big(1 + \frac{\mathscr{R}}{\mathrm{Log}\, n} + O\big((\frac{\mathrm{Log}\,\mathrm{Log}\, n}{\mathrm{Log}\, n})^2 \big) \Big)$$

and (5.116) follows from the condition $|\mathscr{R}| < \frac{\arcsin(\kappa/2)}{\pi}$. $\square$

Theorem 5.28 is the first step of the proof of Theorem 1.4. It is will complemented in §6.3.

6 Minoration of the Mahler measure $\mathrm{M}(\alpha)$ for α a nonreal complex algebraic integer of house $\lceil\alpha\rceil > 1$ close to one

Let α be a nonreal complex algebraic integer, for which $\lceil\alpha\rceil > 1$. Let us assume that the Mahler measure $\mathrm{M}(\alpha)$ is smaller than the smallest Pisot number $\Theta = 1.3247\ldots$. The minimal polynomial $P_\alpha(X) \in \mathbb{Z}[X]$ of α is monic and reciprocal. If $\alpha^{(i)}$ is a conjugate of maximal modulus of α, $\alpha^{(i)}$ is conjugated with $(\alpha^{(i)})^{-1}$; the house $\lceil\alpha\rceil$ of α, resp. its inverse $\lceil\alpha\rceil^{-1}$, is a root of the quadratic equation

$$X^2 - \alpha^{(i)}\overline{\alpha^{(i)}} = 0, \qquad \text{resp. of} \quad X^2 - (\alpha^{(i)})^{-1}(\overline{\alpha^{(i)}})^{-1} = 0.$$

Therefore $\lceil\alpha\rceil$ and $\lceil\alpha\rceil^{-1}$ are real algebraic integers of degree $\leq \deg(\alpha) + 2$ for which $1 < \lceil\alpha\rceil < \Theta = \theta_5^{-1}$. The mapping $\alpha \to \lceil\alpha\rceil, \mathscr{O}_{\overline{\mathbb{Q}}} \to \mathscr{O}_{\overline{\mathbb{Q}}} \cap (1, \infty)$ is not continuous. However, writing $\beta = \lceil\alpha\rceil$, the preceding analytic functions of the Rényi-Parry dynamical system of the β-shift, defined in §4, can be applied once $|\alpha| > 1$ tends to 1^+. The quantities $\mathrm{dyg}(\beta)$, $f_\beta(z)$, $\mathscr{L}_\beta$, $\mathrm{M}(\beta)$, $\mathrm{M}_r(\beta)$ become well-defined so that the minoration of the Mahler measure $\mathrm{M}(\lceil\alpha\rceil)$ is of Dobrowolski type as in Theorem 5.28 in § 5.6, as a function of the dynamical degree $\mathrm{dyg}(\beta)$.

6.1 Fracturability of the minimal polynomial of α by the Parry Upper function at $\lceil\alpha\rceil$

In the following Theorem, which is an extension of Theorem 5.5 and Theorem 5.23, the canonical fracturability of the minimal polynomial $P_\alpha(X)$ by the Parry Upper

function $f_{\lceil\overline{\alpha}\rceil}(z)$ is proved, in full generality, as well as the crucial fact that α is conjugated to its house $\lceil\overline{\alpha}\rceil$. The fundamental consequence (Theorem 6.1 and Theorem 6.5) is the passage from the minoration of $\mathrm{M}_r(\lceil\overline{\alpha}\rceil)$ to that of $\mathrm{M}(\alpha)$ itself, by the identification of the lenticulus $\mathscr{L}_{\lceil\overline{\alpha}\rceil}$ as a lenticulus of conjugates of α itself.

Theorem 6.1. *Let α be a nonreal complex algebraic integer, which satisfies $\lceil\overline{\alpha}\rceil > 1$, $\mathrm{M}(\alpha) < \Theta = 1.3247\ldots$, $\deg(\alpha) \geq 6$. Denote $\beta = \lceil\overline{\alpha}\rceil$. The following formal decomposition of the (monic) minimal polynomial of α*

$$P_\alpha(X) = P^*_\alpha(X) = U_\alpha(X) \times f_\beta(X) \tag{6.1}$$

holds, as a product of the Parry Upper function

$$f_\beta(X) = G_{\mathrm{dyg}(\beta)}(X) + X^{m_1} + X^{m_2} + X^{m_3} + \ldots. \tag{6.2}$$

with $m_0 := \mathrm{dyg}(\beta)$, $m_{q+1} - m_q \geq \mathrm{dyg}(\beta) - 1$ for $q \geq 0$, and the invertible formal series $U_\alpha(X) \in \mathbb{Z}[[X]]$, quotient of P_α by f_β. The specialization $X \to z$ of the formal variable to the complex variable leads to the identity between analytic functions, obeying the Carlson-Polya dichotomy with the house $\lceil\overline{\alpha}\rceil$ of α:

$$P_\alpha(z) = U_\alpha(z) \times f_{\lceil\overline{\alpha}\rceil}(z) \quad \begin{cases} \text{on } \mathbb{C} & \text{if } \lceil\overline{\alpha}\rceil \text{ is a Parry number,} \\ \\ \text{on } |z|<1 & \text{if } \lceil\overline{\alpha}\rceil \text{ is nonParry, with } |z|=1 \\ & \text{as natural boundary for both } U_\alpha \text{ and } f_\beta. \end{cases} \tag{6.3}$$

Assume now $\mathrm{dyg}(\beta) \geq 260$. Let $\mathscr{D}_n$ be the subdomain of the open unit disc defined in Theorem 5.21. Denote $D_{j,n} := \{z \mid |z - z_{j,n}| < \frac{\pi |z_{j,n}|}{n\, a_{\max}}\}$, $j = 1, 2, \ldots, J_n$. Then the domain of holomorphy of $U_\alpha(z) = \frac{P_\alpha(z)}{f_\beta(z)}$ contains the connected domain

$$\Omega_n := \mathscr{D}_n \cup \bigcup_{j=1}^{J_n} \left(D_{j,n} \cup \overline{D_{j,n}}\right) \cup D(\theta_n, \frac{t_{0,n}}{n});$$

$U_\alpha(z)$ does not vanish on the lenticulus $\mathscr{L}_\beta = \{\frac{1}{\beta}\} \cup \bigcup_{j=1}^{J_n}(\{\omega_{j,n}\} \cup \{\overline{\omega_{j,n}}\}) \subset \Omega_n$. For any zero $\omega_{j,n} \in \mathscr{L}_\beta$,

$$U_\alpha(\omega_{j,n}) = \frac{P'_\alpha(\omega_{j,n})}{f'_\beta(\omega_{j,n})} \neq 0, \quad U_\alpha(\overline{\omega_{j,n}}) = \frac{P'_\alpha(\overline{\omega_{j,n}})}{f'_\beta(\overline{\omega_{j,n}})} \neq 0 \quad \text{and } U_\alpha(\frac{1}{\beta}) = \frac{P'_\alpha(\frac{1}{\beta})}{f'_\beta(\frac{1}{\beta})} \neq 0. \tag{6.4}$$

The real algebraic integer $\lceil\overline{\alpha}\rceil$ admits α as conjugate, its minimal polynomial $P_\alpha = P_{\lceil\overline{\alpha}\rceil}$ satisfying

$$P_\alpha(\lceil\overline{\alpha}\rceil) = P_\alpha(\lceil\overline{\alpha}\rceil^{-1}) = P_\alpha(\alpha^{-1}) = P_\alpha(\alpha) = P_\alpha(\overline{\alpha^{-1}}) = P_\alpha(\overline{\alpha}) = 0. \tag{6.5}$$

Proof. Since $\theta_5^{-1} = \Theta > \mathrm{M}(\alpha) \geq \lceil\overline{\alpha}\rceil > 1$, there exists an integer $n \geq 6$ such that $\lceil\overline{\alpha}\rceil$ lies between two successive Perron numbers of the family $(\theta_n^{-1})_{n\geq 5}$, as $\theta_n^{-1} \leq \lceil\overline{\alpha}\rceil <$

θ_{n-1}^{-1}. By Proposition 5.23, the Parry Upper function $f_{\lceil\alpha\rceil}(z)$ at $\lceil\alpha\rceil$ has the form:

$$f_{\lceil\alpha\rceil}(z) = -1 + z + z^n + z^{m_1} + z^{m_2} + z^{m_3} + \ldots \tag{6.6}$$

with $m_0 = n$ and $m_{q+1} - m_q \geq n-1$ for $q \geq 0$. Whether $\lceil\alpha\rceil$ is a Parry number or a non-Parry number is unkown. In any case, by construction, it is such that $f_{\lceil\alpha\rceil}(\lceil\alpha\rceil^{-1}) = 0$. Let us write the Parry Upper function as $f_{\lceil\alpha\rceil}(z) = -1 + \sum_{j\geq 1} t_j z^j$. The zero $\lceil\alpha\rceil^{-1}$ of $f_{\lceil\alpha\rceil}(z)$ is simple since the derivative of $f_{\lceil\alpha\rceil}(z)$ satisfies $f'_{\lceil\alpha\rceil}(\frac{1}{\lceil\alpha\rceil}) = \sum_{j\geq 1} j t_j \lceil\alpha\rceil^{-j+1} > 0$.

Let us show that the formal decomposition (6.1) is always possible. We proceed as in the proof of Theorem 5.5. Indeed, if we put $U_\alpha(X) = -1 + \sum_{j\geq 1} b_j X^j$, and $P_\alpha(X) = 1 + a_1 X + a_2 X^2 + \ldots a_{d-1} X^{d-1} + X^d$, (with $a_j = a_{d-j}$), the formal identity $P_\alpha(X) = U_\alpha(X) \times f_{\lceil\alpha\rceil}(X)$ leads to the existence of the coefficient vector $(b_j)_{j\geq 1}$ of $U_\alpha(X)$, as a function of $(t_j)_{j\geq 1}$ and $(a_i)_{i=1,\ldots,d-1}$, as: $b_1 = -(a_1 + t_1)$, and, for $r = 2, \ldots, d-1$,

$$b_r = -(t_r + a_r - \sum_{j=1}^{r-1} b_j t_{r-j}) \quad \text{with} \quad b_d = -(t_d + 1 - \sum_{j=1}^{d-1} b_j t_{r-j}), \tag{6.7}$$

$$b_r = -t_r + \sum_{j=1}^{r-1} b_j t_{r-j} \quad \text{for } r > d. \tag{6.8}$$

For $j \geq 1$, $b_j \in \mathbb{Z}$, and the integers $b_r, r > d$, are determined recursively by (6.8) from $\{b_0 = -1, b_1, b_2, \ldots, b_d\}$. Every b_j in $\{b_1, b_2, \ldots, b_d\}$ is computed from the coefficient vector of $P_\alpha(X)$ using (6.7), starting by $b_1 = -1 - a_1$. The disc of convergence of $U_\alpha(z)$ has a radius greater than or equal to θ_{n-1} by Theorem 5.5. By Theorem 5.23, assuming $n \geq 260$, the domain of holomorphy of $U_\alpha(z)$ contains Ω_n. By analogy with (5.71) the formal identity holds:

$$P'_\alpha(X) \;=\; U'_\alpha(X) f_\beta(X) \;+\; U_\alpha(X) f'_\beta(X). \tag{6.9}$$

For $j = 1, 2, \ldots, J_n$, since $\omega_{j,n}$ is a simple zero, $f'_\beta(\omega_{j,n}) \neq 0$; specializing X to $\omega_{j,n}$ gives

$$P'_\alpha(\omega_{j,n}) \;=\; U_\alpha(\omega_{j,n}) f'_\beta(\omega_{j,n}), \qquad P'_\alpha(\overline{\omega_{j,n}}) \;=\; U_\alpha(\overline{\omega_{j,n}}) f'_\beta(\overline{\omega_{j,n}});$$

as in the proof of Theorem 5.23, it is easy to show that $|U_\alpha(\omega_{j,n})| \neq +\infty, U_\alpha(\omega_{j,n}) \neq 0$. Similarly, $|U_\alpha(\lceil\alpha\rceil^{-1})| \neq +\infty$ and $U_\alpha(\lceil\alpha\rceil^{-1}) \neq 0$. The zeroes of $\mathscr{L}_{\lceil\alpha\rceil}$ are not singularities of $U_\alpha(z)$. We deduce (6.4) and (6.5).

Definition 6.2. Let α be a nonzero algebraic integer such that $1 < \lceil\alpha\rceil < \Theta = \theta_5^{-1}$. The dynamical degree of α is defined by

$$\mathrm{dyg}(\alpha) := \mathrm{dyg}(\lceil\alpha\rceil);$$

if $n := \mathrm{dyg}(\alpha)$ is ≥ 260, the lenticulus $\mathscr{L}_\alpha$ of conjugates of α is defined by

$$\mathscr{L}_\alpha := \mathscr{L}_{\lceil\overline{\alpha}\rceil} = \{\overline{\omega_{J_n,n}}, \ldots, \overline{\omega_{1,n}}, \lceil\overline{\alpha}\rceil^{-1}, \omega_{1,n}, \ldots, \omega_{J_n,n}\}$$

and the reduced Mahler measure of α by

$$\mathrm{M}_r(\alpha) := \mathrm{M}_r(\lceil\overline{\alpha}\rceil) = \lceil\overline{\alpha}\rceil \prod_{j=1}^{J_n} |\omega_{j,n}|^{-2}.$$

Definition 5.7 can be extended to nonreal complex algebraic integers α as follows.

Definition 6.3. Let α be a nonreal complex algebraic integer such that $\mathrm{M}(\beta) < \Theta$, $\deg(\alpha) \geq 6$, $\lceil\overline{\alpha}\rceil > 1$. The minimal polynomial $P_\alpha(X)$ is said to be fracturable if the power series $U_\alpha(z)$ in (6.3) is not reduced to a constant.

Remark 6.4. Theorem 6.1 extends easily to the three cases (keeping the assumptions $\lceil\overline{\alpha}\rceil > 1$, $\mathrm{M}(\alpha) < \Theta$, and $\mathrm{dyg}(\beta) \geq 260$, in each case): (i) when α is a real algebraic integer which is not > 1, for instance when α is a negative Salem number in which case $\beta = -\alpha > 1$, (ii) when α is real > 1 and that $\alpha \neq \lceil\overline{\alpha}\rceil = \beta$, for instance if α is a totally real algebraic integer > 1, or eventually a partially real algebraic integer > 1, and distinct of its house, (iii) when α is a nonreal complex algebraic integer such $\beta = \lceil\overline{\alpha}\rceil$ admits real Galois conjugates γ satisfying: $1 < \gamma < \beta = \lceil\overline{\alpha}\rceil$.

In the last two cases (ii) and (iii), if we enumerate the distinct real conjugates of β as $1 < \gamma_1 < \gamma_2 < \ldots < \gamma_s < \beta$, $s \geq 1$, the s Parry Upper functions $f_{\gamma_i}(z)$, $i = 1, \ldots, s$, have to be considered together with their respective lenticuli of zeroes, in addition to the Parry Upper function $f_\beta(z)$. It may occur that some zeroes belonging to the lenticuli of zeroes of the functions $f_{\gamma_i}(z)$ lie inside the Rouché disks centered at the points $z_{j,n}$, relative to $f_\beta(z)$. These extra zeroes, distinct of the zeroes $\omega_{j,n}$, are conjugates of α as well. They should be included in the computation of the minorant of the reduced Mahler measure $\mathrm{M}(\alpha)$. This extra collection of zeroes is not really controlled as a continuous function of $\lceil\overline{\alpha}\rceil$ unfortunately. In other terms, in both cases (ii) and (iii), the minorant of the Mahler measure $\mathrm{M}(\alpha)$, obtained below, has certainly to be multiplied by a factor > 1 depending upon the number of real conjugates of the house of α.

For partially or totally real algebraic integers, let us recall Garza's lower bound. Garza [Gza] established the following minoration of the Mahler measure $\mathrm{M}(\alpha)$ for α an algebraic number, different from 0 and ± 1, having a certain proportion of real Galois conjugates: if $\deg(\alpha) = d \geq 1$ and $1 \leq r \leq d$ be the number of real Galois conjugates of α, then

$$\mathrm{M}(\alpha) \geq \left(\frac{2^{1-1/R} + \sqrt{4^{1-1/R} + 4}}{2}\right)^{\frac{r}{2}},$$

where $R := r/d$. An elementary proof of this minoration was given by Höhn [Hhn]. If $r = d$, Garza's bound is Schinzel's bound (7.12) for totally real algebraic integers [HhnSk].

Garza's minorant satisfies $\lim_{d\to\infty} 2^{-r/2}\big(2^{1-d/r} + \sqrt{4^{1-d/r}+4}\big)^{r/2} = 1$, for any r fixed, where the limit 1 is reached "without any discontinuity". In some sense, a better minorant is expected, and Garza's lower bound does not take into account the discontinuity claimed by the Conjecture of Lehmer.

□

6.2 A strange continuity. Asymptotic expansion of the lenticular minorant $\mathrm{M}_r(\lceil\overline{\alpha}\rceil)$

Let α be an algebraic integer such that $\beta = \lceil\overline{\alpha}\rceil$ has dynamical degree $\mathrm{dyg}(\beta) \geq 260$. The continuity of the first nonreal complex root $\omega_{1,n}$ of the lenticulus $\mathscr{L}_\beta$, with β, was proved and studied by Flatto, Lagarias and Poonen [FLP]. By Corollary 4.25 and Theorem 6.1 the others roots of modulus < 1 of the Parry Upper function $f_{\lceil\overline{\alpha}\rceil}(z)$ which are conjugates of α are continuous functions of $\beta = \lceil\overline{\alpha}\rceil$. These facts suggest the conjecture that the (true) Mahler measure $\mathrm{M}(\alpha)$ is a continuous function of the house $\lceil\overline{\alpha}\rceil$ of α. On the contrary, the nonderivability of the function $\beta = \lceil\overline{\alpha}\rceil \to \omega_{1,\mathrm{dyg}(\beta)}$ conjectured in [FLP] would suggest that the (true) Mahler measure $\mathrm{M}(\alpha)$ is not derivable as a function of $\beta = \lceil\overline{\alpha}\rceil$, in general.

Theorem 6.5. *Let α be an algebraic integer such that $n = \mathrm{dyg}(\lceil\overline{\alpha}\rceil) \geq 260$. The asymptotic expansion of the minorant $L_r(\lceil\overline{\alpha}\rceil)$ of $\mathrm{Log}\,\mathrm{M}_r(\alpha)$ is*

$$L_r(\lceil\overline{\alpha}\rceil) = \mathrm{Log}\,\Lambda_r\mu_r + \frac{\mathscr{R}}{\mathrm{Log}\,n} + O\Big(\big(\frac{\mathrm{Log}\,\mathrm{Log}\,n}{\mathrm{Log}\,n}\big)^2\Big), \quad \textit{with} \quad |\mathscr{R}| < \frac{\arcsin(\kappa/2)}{\pi} \tag{6.10}$$

and $\mathscr{R}$ depending upon $\lceil\overline{\alpha}\rceil$ and n.

Proof. Using Definition 6.2, with $\beta = \lceil\overline{\alpha}\rceil$, we readily deduce the result from Theorem 5.27 and Theorem 5.28. □

As in the proof of Theorem 6.1 the minoration of the lenticular Mahler measure of β follows from (6.10):

$$\mathrm{M}_r(\beta) \geq \Lambda_r\mu_r\big(1 + \frac{\mathscr{R}}{\mathrm{Log}\,n} + O\Big(\big(\frac{\mathrm{Log}\,\mathrm{Log}\,n}{\mathrm{Log}\,n}\big)^2\Big), \quad \text{with } |\mathscr{R}| < \frac{\arcsin(\kappa/2)}{\pi}. \tag{6.11}$$

6.3 A Dobrowolski type minoration with the dynamical degree of $\lceil\overline{\alpha}\rceil$ - Proof of Theorem 1.4

Using Definition 6.2, with $\beta = \lceil\overline{\alpha}\rceil$, we deduce (1.11) from Theorem 6.5 and (6.11). In the case where α is the conjugate of a Perron number $\theta_n^{-1}, n \geq 260$, the minorant in (1.11) takes much higher values and is already given in §3 (Theorem 3.16).

6.4 Proof of the Conjecture of Lehmer (Theorem 1.2)

Let $\alpha \neq 0$ be an algebraic integer which is not a root of unity. Since $\mathrm{M}(\alpha) = \mathrm{M}(\alpha^{-1})$ there are three cases to be considered:

(i) the house of α satisfies $\lceil\overline{\alpha}\rceil \geq \theta_5^{-1}$,

(ii) the dynamical degree of α satisfies: $6 \leq \mathrm{dyg}(\alpha) < 260$,

(iii) the dynamical degree of α satisfies: $\mathrm{dyg}(\alpha) \geq 260$.

In the first case, $\mathrm{M}(\alpha) \geq \theta_5^{-1} \geq \theta_{259}^{-1}$. In the second case, $\mathrm{M}(\alpha) \geq \theta_{259}^{-1}$. In the third case, the Dobrowolski type inequality (1.11) gives the following lower bound of the Mahler measure

$$\mathrm{M}(\alpha) \geq \Lambda_r \mu_r - \frac{\Lambda_r \mu_r \arcsin(\kappa/2)}{\pi \operatorname{Log}(\mathrm{dyg}(\alpha))} \geq \Lambda_r \mu_r - \frac{\Lambda_r \mu_r \arcsin(\kappa/2)}{\pi \operatorname{Log}(259)}, = 1.14843\ldots$$

by Proposition 5.12 and Theorem 5.26. This lower bound is numerically greater than $\theta_{259}^{-1} = 1.016126\ldots$. Therefore, in any case, the lower bound θ_{259}^{-1} of $\mathrm{M}(\alpha)$ holds true. We deduce the claim.

6.5 Proof of the Conjecture of Schinzel-Zassenhaus (Theorem 1.3)

Proposition 6.6. *Let α be an algebraic integer such that* $\mathrm{dyg}(\alpha) \geq 260$*. The degree* $\deg(\alpha)$ *of α is related to its dynamical degree* $\mathrm{dyg}(\alpha)$ *by*

$$\mathrm{dyg}(\alpha)\Big(\frac{2\arcsin\big(\frac{\kappa}{2}\big)}{\pi}\Big) + \Big(\frac{2\kappa \operatorname{Log}\kappa}{\pi\sqrt{4-\kappa^2}}\Big) \;\leq\; \deg(\alpha). \tag{6.12}$$

Proof. By Theorem 5.15 and Theorem 5.23 the number of zeroes in the lenticulus $\mathscr{L}_\alpha$ is $1+2J_n$; these zeroes are all conjugates of α. The total number of conjugates of α is the degree $\deg(\alpha)$ of the minimal polynomial P_α. By Proposition 5.12, with $n := \deg(\alpha)$,

$$1 + 2J_n = \frac{2n}{\pi}\big(\arcsin\big(\frac{\kappa}{2}\big)\big) + \Big(\frac{2\kappa \operatorname{Log}\kappa}{\pi\sqrt{4-\kappa^2}}\Big) + \big(1 + \frac{1}{n}O\big(\big(\frac{\operatorname{Log}\operatorname{Log} n}{\operatorname{Log} n}\big)^2\big)\big).$$

The inequality (6.12) follows.

□

Theorem 6.7. *Let α be a nonzero algebraic integer which is not a root of unity. Then*

$$\overline{|\alpha|} \geq 1 + \frac{c}{\deg(\alpha)} \tag{6.13}$$

with $c = \theta_{259}^{-1} - 1 = 0.016126\ldots$.

Proof. There are two cases: either (i) $\overline{|\alpha|} \geq \theta_{259}^{-1}$, or (ii) $n \geq 260$.

(i) If $\overline{|\alpha|} \geq \theta_{259}^{-1}$, then, whatever the degree $\deg(\alpha) \geq 1$,

$$\overline{|\alpha|} \geq 1 + \frac{(\theta_{259}^{-1} - 1)}{\deg(\alpha)}.$$

(ii) The minoration of the house $\beta = \overline{|\alpha|}$ can easily be obtained as a function of the dynamical degree of α. Let $n = \mathrm{dyg}(\beta)$ and assume $n \geq 260$. By definition $\theta_n^{-1} \leq \beta < \theta_{n-1}^{-1}$. Theorem 1.8 in [VG6] (cf also [VG6] §5.3) implies

$$\beta = \overline{|\alpha|} \;\geq\; \theta_n^{-1} \;\geq\; 1 + \frac{(\mathrm{Log}\, n)(1 - \frac{\mathrm{Log}\,\mathrm{Log}\, n}{\mathrm{Log}\, n})}{n}. \tag{6.14}$$

From Proposition 6.6,

$$\frac{1}{n} = \frac{1}{\mathrm{dyg}(\beta)} \geq \frac{2 \arcsin(\kappa/2)}{\pi \,\deg(\alpha)} \left(1 + \frac{\kappa \mathrm{Log}\, \kappa}{n \arcsin(\kappa/2) \sqrt{4 - \kappa^2}}\right). \tag{6.15}$$

The function $\frac{\mathrm{Log}\, x - \mathrm{Log}\,\mathrm{Log}\, x}{\mathrm{Log}\, x} \left(1 + \frac{\kappa \mathrm{Log}\, \kappa}{x \arcsin(\kappa/2) \sqrt{4-\kappa^2}}\right)$ is increasing for $x \geq 260$. From (6.14) and (6.15) we deduce

$$\overline{|\alpha|} \geq 1 + \frac{\tilde{c}}{\deg(\alpha)}$$

with $\tilde{c} = \frac{2}{\pi} \frac{\mathrm{Log}\, 260 - \mathrm{Log}\,\mathrm{Log}\, 260}{\mathrm{Log}\, 260} \left(\arcsin(\kappa/2) + \frac{\kappa \mathrm{Log}\, \kappa}{260 \sqrt{4-\kappa^2}}\right) = 0.0375522\ldots$.

From (i) and (ii), we deduce that (6.13) holds with $c = \min\{\tilde{c}, (\theta_{259}^{-1} - 1)\} = (\theta_{259}^{-1} - 1) = 0.016126\ldots$ for every nonzero algebraic integer α which is not a root of unity. □

7 Salem numbers, totally real algebraic numbers, Bogomolov property

The set of Pisot numbers admits the minorant Θ by a result of Siegel [Si]. We extend this result in the sequel as a consequence of Theorem 7.3: in fact Theorem 7.3 implies boundedness from below to (i) the set of Salem numbers (§ 7.2), (ii) the set of totally real algebraic numbers in terms of the Weil height (§ 7.3).

7.1 Existence and localization of the first nonreal root of the Parry Upper function $f_\beta(z)$ of modulus < 1 in the cusp of the fractal of Solomyak

Theorem 7.1. *Let $n \geq 32$. Denote by $C_{1,n} := \{z \mid |z - z_{1,n}| = \frac{\pi |z_{1,n}|}{n\, a_{\max}}\}$ the circle centered at the first root $z_{1,n}$ of $G_n(z) = -1 + z + z^n$. Then the condition of Rouché*

$$\frac{|z|^{2n-1}}{1 - |z|^{n-1}} \;<\; |-1 + z + z^n|, \qquad \text{for all } z \in C_{1,n}, \tag{7.1}$$

holds true.

Proof. Let $a \geq 1$ and $n \geq 18$. Denote by $\varphi := \arg(z_{1,n})$ the argument of the first root $z_{1,n}$ (in $\mathrm{Im}(z) > 0$). Since $-1 + z_{1,n} + z_{1,n}^n = 0$, we have $|z_{1,n}|^n = |-1 + z_{1,n}|$. Let us write $z = z_{1,n} + \frac{\pi |z_{1,n}|}{n\, a} e^{i\psi} = z_{1,n}(1 + \frac{\pi}{an} e^{i(\psi - \varphi)})$ the generic element belonging to $C_{1,n}$, with $\psi \in [0, 2\pi]$. Let $X := \cos(\psi - \varphi)$. Let us show that if the inequality (7.1) of Rouché holds true for $X = +1$, then it holds true for all $X \in [-1, +1]$, that is for every argument $\psi \in [0, 2\pi]$, i.e. for every $z \in C_{1,n}$. As in the proof of Theorem 5.9,

$$\left|1 + \frac{\pi}{a\, n} e^{i(\psi - \varphi)}\right|^n = \exp\Big(\frac{\pi X}{a}\Big) \times \left(1 - \frac{\pi^2}{2a^2\, n}(2X^2 - 1) + O(\frac{1}{n^2})\right)$$

and

$$\arg\left(\Big(1 + \frac{\pi}{a\, n} e^{i(\psi - \varphi)}\Big)^n\right) = sgn(\sin(\psi - \varphi)) \times \left(\frac{\pi \sqrt{1 - X^2}}{a}[1 - \frac{\pi X}{a\, n}] + O(\frac{1}{n^2})\right).$$

Moreover,

$$\left|1 + \frac{\pi}{a\, n} e^{i(\psi - \varphi)}\right| = \left|1 + \frac{\pi}{a\, n}(X \pm i\sqrt{1 - X^2})\right| = 1 + \frac{\pi X}{a\, n} + O(\frac{1}{n^2}).$$

with

$$\arg(1 + \frac{\pi}{a\, n} e^{i(\psi - \varphi)}) = sgn(\sin(\psi - \varphi)) \times \frac{\pi \sqrt{1 - X^2}}{a\, n} + O(\frac{1}{n^2}).$$

For all $n \geq 18$, from Proposition 3.9, we have

$$|z_{1,n}| = 1 - \frac{\mathrm{Log}\, n - \mathrm{Log}\,\mathrm{Log}\, n}{n} + \frac{1}{n} O\left(\frac{\mathrm{Log}\,\mathrm{Log}\, n}{\mathrm{Log}\, n}\right). \tag{7.2}$$

from which we deduce the following equality, up to $O(\frac{1}{n})$ - terms,

$$|z_{1,n}| \left|1 + \frac{\pi}{a\, n} e^{i(\psi - \varphi)}\right| = |z_{1,n}|.$$

Then the left-hand side term of (7.1) is

$$\frac{|z|^{2n-1}}{1-|z|^{n-1}} = \frac{|-1+z_{1,n}|^2 \left|1+\frac{\pi}{an}e^{i(\psi-\varphi)}\right|^{2n}}{|z_{1,n}|\,\left|1+\frac{\pi}{an}e^{i(\psi-\varphi)}\right| - |-1+z_{1,n}|\,\left|1+\frac{\pi}{an}e^{i(\psi-\varphi)}\right|^{n}}$$

$$= \frac{|-1+z_{1,n}|^2 \left(1-\frac{\pi^2}{an}(2X^2-1)\right)\exp\left(\frac{2\pi X}{a}\right)}{|z_{1,n}|\left|1+\frac{\pi}{an}e^{i(\psi-\varphi)}\right| - |-1+z_{1,n}|\,\left(1-\frac{\pi^2}{2an}(2X^2-1)\right)\exp\left(\frac{\pi X}{a}\right)} \tag{7.3}$$

up to $\frac{1}{n}O\left(\frac{\mathrm{Log}\,\mathrm{Log}\, n}{\mathrm{Log}\, n}\right)$ -terms (in the terminant). The right-hand side term of (7.1) is

$$|-1+z+z^n| = \left|-1+z_{1,n}\left(1+\frac{\pi}{na}e^{i(\psi-\varphi)}\right)+z_{1,n}^n\left(1+\frac{\pi}{na}e^{i(\psi-\varphi)}\right)^n\right|$$

$$= \Bigg|-1+z_{1,n}\left(1\pm i\frac{\pi\sqrt{1-X^2}}{an}\right)\left(1+\frac{\pi X}{an}\right)+(1-z_{1,n})\left(1-\frac{\pi^2}{2a^2 n}(2X^2-1)\right)$$

$$\times\exp\left(\frac{\pi X}{a}\right)\exp\left(\pm i\left(\frac{\pi\sqrt{1-X^2}}{a}\left[1-\frac{\pi X}{an}\right]\right)\right)+O\left(\frac{1}{n^2}\right)\Bigg| \tag{7.4}$$

Let us consider (7.3) and (7.4) at the first order for the asymptotic expansions, i.e. up to $O(1/n)$ - terms instead of up to $O(\frac{1}{n}(\mathrm{Log}\,\mathrm{Log}\, n/\mathrm{Log}\, n))$ - terms or $O(1/n^2)$ - terms. (7.3) becomes:

$$\frac{|-1+z_{1,n}|^2\exp\left(\frac{2\pi X}{a}\right)}{|z_{1,n}|-|-1+z_{1,n}|\exp\left(\frac{\pi X}{a}\right)}$$

and (7.4) is equal to:

$$|-1+z_{1,n}|\left|1-\exp\left(\frac{\pi X}{a}\right)\exp\left(\pm i\frac{\pi\sqrt{1-X^2}}{a}\right)\right|$$

and is independent of the sign of $\sin(\psi-\varphi)$. Then the inequality (7.1) is equivalent to

$$\frac{|-1+z_{1,n}|^2\exp\left(\frac{2\pi X}{a}\right)}{|z_{1,n}|-|-1+z_{1,n}|\exp\left(\frac{\pi X}{a}\right)} < |-1+z_{1,n}|\left|1-\exp\left(\frac{\pi X}{a}\right)\exp\left(\pm i\frac{\pi\sqrt{1-X^2}}{a}\right)\right|, \tag{7.5}$$

and (7.5) to

$$\frac{|-1+z_{1,n}|}{|z_{1,n}|} < \frac{\left|1-\exp\left(\frac{\pi X}{a}\right)\exp\left(i\frac{\pi\sqrt{1-X^2}}{a}\right)\right|\exp\left(\frac{-\pi X}{a}\right)}{\exp\left(\frac{\pi X}{a}\right)+\left|1-\exp\left(\frac{\pi X}{a}\right)\exp\left(i\frac{\pi\sqrt{1-X^2}}{a}\right)\right|} = \kappa(X,a). \tag{7.6}$$

The right-hand side function $\kappa(X,a)$ is a function of (X,a), on $[-1,+1]\times[1,+\infty)$. which is strictly decreasing for any fixed a, and reaches its minimum at $X=1$; this minimum is always strictly positive. Consequently the inequality of Rouché (7.1) will be satisfied on $C_{1,n}$ once it is satisfied at $X=1$, as claimed.

Hence, up to $O(1/n)$-terms, the Rouché condition (7.6), for any fixed a, will be satisfied (i.e. for any $X\in[-1,+1]$) by the set of integers $n=n(a)$ for which $z_{1,n}$ satisfies:

$$\frac{|-1+z_{1,n}|}{|z_{j,n}|}<\kappa(1,a)=\frac{\left|1-\exp\left(\frac{\pi}{a}\right)\right|\exp\left(\frac{-\pi}{a}\right)}{\exp\left(\frac{\pi}{a}\right)+\left|1-\exp\left(\frac{\pi}{a}\right)\right|}, \tag{7.7}$$

equivalently, from Proposition 3.14,

$$\frac{\operatorname{Log} n-\operatorname{Log}\operatorname{Log} n}{n}<\frac{\kappa(1,a)}{1+\kappa(1,a)}. \tag{7.8}$$

In order to obtain the largest possible range of values of n, the value of $a\geq 1$ has to be chosen such that $a\to\kappa(1,a)$ is maximal in (7.8) (Figure 2). In the proof of Theorem 5.9 we have seen that the function $a\to\kappa(1,a)$ reaches its maximum $\kappa(1,a_{\max}):=0.171573\ldots$ at $a_{\max}=5.8743\ldots$. We take $a=a_{\max}$.

The slow decrease of the functions of the variable n involved in the terminants when n tends to infinity, as a factor of uncertainty on (7.8), has to be taken into account in (7.8). It amounts to check numerically whether (7.1) is satisfied for the small values $18\leq n\leq 100$ for $a=a_{\max}$, or not. Indeed, for the large enough values of n, the inequality (7.8) is satisfied since $\lim_{n\to\infty}\frac{\operatorname{Log} n-\operatorname{Log}\operatorname{Log} n}{n}=0$. On the computer, the critical threshold of $n=32$ is easily calculated, with $(\operatorname{Log} 32-\operatorname{Log}\operatorname{Log} 32)/32=0.0694628\ldots$.Then

$$\frac{\operatorname{Log} n-\operatorname{Log}\operatorname{Log} n}{n}<\frac{\kappa(1,a_{\max})}{1+\kappa(1,a_{\max})}=0.146447\ldots \qquad \text{for all } n\geq 32.$$

Let us note that the last inequality also holds for some values of n less than 32. □

Corollary 7.2. *Let $\beta>1$ be any algebraic number of dynamical degree $\operatorname{dyg}(\beta)\geq 32$. Then the Parry Upper function $f_\beta(z)$ admits a simple zero $\omega_{1,n}$ (of modulus <1) in the open disc $D(z_{1,n},\frac{\pi|z_{1,n}|}{n\,a_{\max}})$.*

Proof. The polynomial $G_n(z)$ has simple roots. Since (7.1) is satisfied, the Theorem of Rouché states that $f_\beta(z)$ and $G_n(z)=-1+z+z^n$ have the same number of roots, counted with multiplicities, in the open disc $D(z_{1,n},\frac{\pi|z_{1,n}|}{n\,a_{\max}})$, giving the existence of an unique zero $\omega_{1,n}$. □

Let us prove that the first zero $\omega_{1,n}$ of $f_\beta(z)$ is a zero of the minimal polynomial of β, which lies in the cusp of the fractal of Solomyak (§ 4.2.2). In the following theorem the minimal polynomial need not be monic.

Theorem 7.3. *Let $\beta > 1$ be any algebraic number of dynamical degree* $\mathrm{dyg}(\beta) \geq 32$. *Then the minimal polynomial $P_\beta(X)$ and the Parry Upper function $f_\beta(z)$ satisfy the canonical identity of the complex variable z*

$$P_\beta(z) \;=\; U_\beta(z) \times f_\beta(z) \tag{7.9}$$

where $U_\beta(z) = \frac{P_\beta(z)}{f_\beta(z)} \in \mathbb{Z}[[z]]$ is holomorphic on the open disc $D_{1,n} = \{z \mid |z - z_{1,n}| < \frac{\pi |z_{1,n}|}{n\, a_{\max}}\}$ having no zero on this disc. Moreover, if $\omega_{1,n}$ is the unique zero of $f_\beta(z)$ inside this disc, we have:

$$U_\beta(\omega_{1,n}) = \frac{P'_\beta(\omega_{1,n})}{f'_\beta(\omega_{1,n})}. \tag{7.10}$$

The zero $\omega_{1,n} = \omega_{1,n}(\beta)$ of $f_\beta(z)$ is a nonreal complex zero of modulus < 1 of the minimal polynomial $P_\beta(z)$, and a continous function of β.

Proof. The analytic function $f_\beta(z)$ obeys the Carlson-Polya dichotomy (Bell and Chen [BCn], Carlson [C] [C2], Dienes [Dis], Polya [Pl], Szegő [Szo]), as already mentioned in Theorem 5.23 when $n \geq 260$. It is defined as an holomorphic function on the open unit disc. Therefore, whatever the Rényi-Parry dynamics of β, i.e. if β is a Parry number or a nonParrynumber, $f_\beta(z)$ is holomorphic on the open disc $D_{1,n}$ this disc being included in $|z| < 1$. The function $f_\beta(z)$ admits only one zero in the disc $D_{1,n}$, which is simple, by Corollary 7.2. This zero $\omega_{1,n}$ satisfies: $|\omega_{1,n} - z_{1,n}| < \frac{\pi |z_{1,n}|}{n\, a_{\max}}$. The minimal polynomial P_β is monic or not; in both cases the function $U_\beta(z) = \frac{P_\beta(z)}{f_\beta(z)}$ belongs to $\mathbb{Z}[[z]]$ and is analytic on the open unit disc; the unit circle $|z| = 1$ is natural boundary if and only if β is a nonParry number. Inside the open unit disc, the function $U_\beta(z)$ eventually admits poles and zeroes. Let us show that $U_\beta(z)$ has no pole inside $D_{1,n}$. Indeed, since $P_\beta(z)$ has only simple roots, that $P_\beta(z) = U_\beta(z) \times f_\beta(z)$ and that $f_\beta(z)$ has an unique zero in $D_{1,n}$, the function $U_\beta(z)$ either has a simple pole at $\omega_{1,n}$ or does not vanish at $\omega_{1,n}$. For any $z \in D_{1,n}$, $z \neq \omega_{1,n}$, $U_\beta(z)$ is holomorphic at z. Let us show that the function $U_\beta(z) = \frac{P_\beta(z)}{f_\beta(z)}$ has no pole at $\omega_{1,n}$. Deriving the identity $P_\beta(X) = U_\beta(X) \times f_\beta(X)$ and specializing the formal variable X to the complex variable z gives:

$$P'_\beta(z) = U'_\beta(z) \times f_\beta(z) + U_\beta(z) \times f'_\beta(z).$$

Therefore,

$$P'_\beta(\omega_{1,n}) = U_\beta(\omega_{1,n}) \times f'_\beta(\omega_{1,n}),$$

from which we deduce (7.10) and the identification of the zero $\omega_{1,n}$ of $f_\beta(z)$ in $D_{1,n}$ as a conjugate of β. Of course, the minimal polynomial may possibly admit other roots inside $D_{1,n}$; in this case these other roots would be zeroes of the function $U_\beta(z)$, not of $f_\beta(z)$. Since $|\omega_{1,n}| < 1$, that $z_{1,n}$ is a nonreal complex number and the radius

$\pi|z_{1,n}|/(n_{\max})$ of the disc $D_{1,n}$ is small enough, the polynomial $P_\beta(z)$ admits the nonreal complex root $\omega_{1,n}$ inside the open unit disc. The map $\beta \to \omega_{1,n}(\beta)$ is continuous by Corollary 4.25 in § 4.4. In Flatto, Lagarias and Poonen [FLP] the first root $\omega_{1,n}$, and its continuity with β, was studied as a zero of $f_\beta(z)$; here, after identifying it as a Galois conjugate of β, $\omega_{1,n}$ is a zero of the minimal polynomial $P_\beta(z)$. In other terms, the normal closure of $\mathbb{Q}(\beta)$ admits the couple of nonreal complex embeddings $\beta \to (\omega_{1,n}, \overline{\omega_{1,n}})$ continuous with β. □

7.2 A lower bound for the set of Salem numbers. Proof of Theorem 1.5

Negative and (positive) Salem numbers occur in number theory, e.g. for graphs or integer symmetric matrices in [MS] [MS2] [MS3], and in other domains (cf § 2.2), like Alexander polynomials of links of the variable "$-x$", e.g. in Theorem 2.36 [Ha]. The closed set S of Pisot numbers and its successive derivatives $S^{(i)}$, were extensively studied by Dufresnoy and Pisot, and their students (Amara, ...), by means of compact families of meromorphic functions, following ideas of Schur [B-S]. On the contrary the set of Salem numbers is badly known. Association equations between Pisot numbers and Salem numbers were used to study these numbers [B-S]. Salem (1944) (Samet [Set]) proved that every Salem number is the quotient of two Pisot numbers. In the same direction association equations between Salem numbers and (generalized) Garsia numbers (Hare and Panju [HPu]) were recently studied, using interlacing theory on the unit circle [GdVG] [Los2] [Los4] [Los5] [MS3].

As counterpart, Salem numbers are linked to units: they are given by closed formulas from Stark units in Chinburg [Cg] [Cg2], exceptional units in Silverman [Sn3]. From [Cg2] they are related to relative regulators of number fields [A5] [A6] [CMcK] [CaFn] [GH].

Proof of Theorem 1.5: Assume that β is a Salem number of dynamical degree $n = \mathrm{dyg}(\beta) \geq 32$. Its minimal polynomial $P_\beta(X)$ would admit β, $1/\beta$ as real roots, the remaining roots being on the unit circle, as (nonreal) complex-conjugated pairs. By Theorem 7.3 it would admit the pair of nonreal roots $(\omega_{1,n}, \overline{\omega_{1,n}})$ as well, strictly inside the open unit disc. This fact is impossible. We deduce that $\beta > \theta_{31}^{-1} = 1.08545\ldots$.

7.3 Totally real algebraic integers, Bogomolov property. Proof of Theorem 1.6

Let $\mathbb{L}$ be a totally real algebraic number field, or a CM field (a totally complex quadratic extension of a totally real number field). Then, for any nonzero algebraic integer $\alpha \in \mathbb{L}$, of degree d, not being a root of unity, Schinzel [Sc2] obtained the

minoration

$$\mathrm{M}(\alpha) \geq \theta_2^{-d/2} = \left(\frac{1+\sqrt{5}}{2}\right)^{d/2}. \tag{7.11}$$

More precisely, if $H(X) \in \mathbb{Z}[X]$ is monic with degree d, $H(0) = \pm 1$ and $H(-1)H(1) \neq 0$, and if the zeroes of H are all real, then

$$\mathrm{M}(H) \geq \Big(\frac{1+\sqrt{5}}{2}\Big)^{d/2} \tag{7.12}$$

with equality if and only if $H(X)$ is a power of $X^2 - X - 1$. Bertin [Bn] improved Schinzel's minoration (7.12) for the algebraic integers α, of degree d, of norm $N(\alpha)$, which are totally real, as

$$\mathrm{M}(\alpha) \geq \max\big\{\theta_2^{-d/2}, \sqrt{N(\alpha)}\,\theta_2^{-\frac{d}{2|N(\alpha)|^{1/d}}}\big\}.$$

The totally real algebraic numbers form a subfield, denoted by $\mathbb{Q}^{tr}$, in $\overline{\mathbb{Q}} \cap \mathbb{R}$. Following [Bn], the natural extension of a Salem number is a ν-Salem number, intermediate between Salem numbers and totally real algebraic numbers. Let us define a ν-Salem as an algebraic integer α having ν conjugates outside $\{|z| \geq 1\}$ and at least one conjugate $\alpha^{(q)}$ satisfying $|\alpha^{(q)}| = 1$; denote by $2\nu + 2k$ its degree. Such an algebraic integer is totally real in the sense that its conjugates of modulus > 1 are all real, and then

$$\mathrm{M}(\alpha) \geq \theta_2^{-\frac{\nu}{2^{k/\nu}}}.$$

Further, extending Pisot numbers, lower bounds of $\mathrm{M}(\alpha)$ were obtained by Zaimi [Zi] [Zi2] when α is a K-Pisot number. Rhin [Rn], following Zaimi (cf references in [Rn]), obtained minorations of $\mathrm{M}(\alpha)$ for totally positive algebraic integers α as functions of the discriminant $\mathrm{disc}(\alpha)$. Let K be an algebraic number field and α an algebraic integer of minimal polynomial R over K; by definition [BeM] α is K-Pisot number if, for any embedding $\sigma : K \to \mathbb{C}$, $\sigma(R_K)$ admits only one root of modulus > 1 and no root of modulus 1. Denote by Δ the discriminant of K. Lehmer's problem and small discriminants were studied by Mahler (1964), Bertrand [Bed], Matveev [Mv2], Rhin [Rn]. For any K-Pisot number α, Zaimi [Zi] [Zi2] showed

$$\mathrm{M}(\alpha) \geq \frac{\sqrt{\Delta}}{2} \qquad K \text{ quadratic}, \tag{7.13}$$

$$\mathrm{M}(\alpha) \geq \frac{\Delta^{1/4}}{\sqrt{6}} \qquad K \text{ cubic and totally real}. \tag{7.14}$$

Other minorations of totally positive algebraic integers were obtained by Mu and Wu [MuWu]. Denote $\mathbb{Z}^{tr} := \mathbb{Q}^{tr} \cap \mathscr{O}_{\overline{\mathbb{Q}}}$. Because the degree d of the algebraic number commonly appears in the exponent of the lower bounds of the Mahler measure,

the (absolute logarithmic) Weil height h is more adapted than the Mahler measure. Schinzel's bound, originally concerned with the algebraic integers in $\mathbb{Z}^{tr}$, reads:

$$\alpha \in \mathbb{Z}^{tr}, \alpha \neq 0, \neq \pm 1 \quad \Rightarrow \quad h(\alpha) \geq h\big(\theta_2^{-1}\big) = \frac{1}{2}\mathrm{Log}\,(\frac{1+\sqrt{5}}{2}) = 0.2406059\ldots$$

Smyth [Sy3] [Sy4] proved that the set

$$\{\exp(h(\alpha)) \mid \alpha \text{ totally real algebraic integer}, \alpha \neq 0, \neq \pm 1\}$$

is everywhere dense in $(1.31427\ldots, \infty)$; in other terms

$$\liminf_{\alpha \in \mathbb{Z}^{tr}} h(\alpha) \;\leq\; \mathrm{Log}\,(1.31427\ldots) = 0.27328\ldots$$

Flammang [Fg] completed Smyth's results by showing

$$\liminf_{\alpha \in \mathbb{Z}^{tr}} h(\alpha) \;\geq\; \frac{1}{2}\mathrm{Log}\,(1.720566\ldots) = 0.271327\ldots$$

with exactly 6 isolated points in the interval $(0, 0.271327\ldots)$, the smallest one being Schinzel's bound $0.2406059\ldots$ In fact, passing from algebraic integers to algebraic numbers lead to various smaller minorants of $h(\alpha)$: for instance $(\mathrm{Log}\,5)/12 = 0.134119\ldots$ by Amoroso and Dvornicich [AD] for any nonzero $\alpha \in \mathbb{L}$ which is not a root of unity, where $\mathbb{L}/\mathbb{Q}$ is an abelian extension of number fields, or, by Ishak, Mossinghoff, Pinner and Wiles [IMPW], for nonzero $\alpha \in \mathbb{Q}(\xi_m)$, not being a root of unity,

(i) $h(\alpha) \geq 0.155097\ldots$, for 3 not dividing m,

(ii) $h(\alpha) \geq 0.166968\ldots$, for 5 not dividing m, unless $\alpha = \alpha_0^{\pm 1}\zeta$, with ζ a root of unity, whence $h(\alpha) \geq (\mathrm{Log}\,7)/12 = 0.162159\ldots$, α_0 being a root of $7X^{12} - 13X^6 + 7$,

(iii) $h(\alpha) \geq 0.162368\ldots$, for 7 not dividing m.

(cf also [AD] [AZ2] [Gza] [Gza2] [IMPW] [Pyr] for other results). For totally real numbers α, Fili and Miner [FMr], using results of Favre and Rivera-Letelier [FLr] on the equidistribution of points of small Weil height, obtained the limit infimum of the height

$$\liminf_{\alpha \in \mathbb{Q}^{tr}} h(\alpha) \;\geq\; \frac{140}{3}\left(\frac{1}{8} - \frac{1}{6\pi}\right)^2 = 0.120786\ldots$$

Bombieri and Zannier [BriZ] have recently introduced the concept of "Bogomolov property", by analogy with the "Bogomolov Conjecture". Let us recall it. Assuming a fixed choice of embedding $\overline{\mathbb{Q}} \to \mathbb{C}$, a field $\mathbb{K} \subset \overline{\mathbb{Q}}$ is said to possess the Bogomolov property relative to h is and only if $h(\alpha)$ is zero or bounded from below by a positive constant for all $\alpha \in \mathbb{K}$. The search of small Weil's heights is important [AD] [ANo] Choi [Chi]. Every number field has the Bogomolov property relative to h by Northcott's theorem [Swt] [Swt2]. Other fields are known to possess the Bogomolov property: (i) $\mathbb{Q}^{tr}$ [Sc2], (ii) finite extensions of the maximal abelian extensions of number fields [AZ] [AZ2], (iii) totally p-adic fields [BriZ], i.e. for algebraic numbers all of whose conjugates lie in $\mathbb{Q}_p$, (iv) $\mathbb{Q}(E_{tors})$ for $E/\mathbb{Q}$ an elliptic curve [Hgr].

Our result is that the Weil height of any nonzero totally real algebraic number $\neq \pm 1$ is bounded from below as stated in Theorem 1.6. The proof below is actually another proof of Schinzel's theorem which states (in Bombieri Zannier's notation) that the field $\mathbb{Q}^{tr}$ has the Bogomolov property relative to h.

Proof of Theorem 1.6:
(i) Let α be a totally real algebraic integer $\neq 0, \neq \pm 1$, $\deg(\alpha) \geq 1$. Assume that its minimal polynomial $P_\alpha(x) = \prod_{i=1}^{\deg(\alpha)} (x - \alpha^{(i)})$ is totally positive, i.e. all its roots are real and positive. The Mahler measure $\mathrm{M}(P_\alpha)$ of the minimal polynomial of α is equal to the Mahler measure $\mathrm{M}(P_\alpha^*)$ of its reciprocal polynomial. If P_α is reciprocal, then the number of conjugates $\alpha^{(i)} > 1$ is equal to the number of conjugates $\alpha^{(i)}$ which are in $(0,1)$. Denote by β the smallest conjugate of α which is > 1. The following inequality holds:

$$\mathrm{M}(\alpha) \geq \beta^{\deg(\alpha)/2}.$$

We now apply Theorem 7.3 to β. The conjugates of β are the conjugates of α. They all lie on the real line. If we assume $n = \mathrm{dyg}(\beta) \geq 32$, we arrive at a contradiction since P_α would admit the nonreal complex $\omega_{1,n}$ as zero. Therefore $\beta > \theta_{31}^{-1}$ and

$$\frac{\mathrm{Log}\,\mathrm{M}(\alpha)}{\deg(\alpha)} = h(\alpha) > \frac{1}{2}\,\mathrm{Log}\,\theta_{31}^{-1} = 0.04...$$

If P_α is not reciprocal and that the number x of conjugates of α which are > 1 is $\geq \deg(\alpha)/2$, we denote by β the smallest conjugate of α which is > 1. Then $\mathrm{M}(\alpha) \geq \beta^x \geq \beta^{\deg(\alpha)/2}$ and $h(\alpha) > \frac{1}{2}\,\mathrm{Log}\,\theta_{31}^{-1} = 0.04...$ as above with the same argument. If P_α is not reciprocal and that $x < \deg(\alpha)/2$, then we consider P_α^*. Then the number of conjugates of α^{-1} which are > 1 is $\geq \deg(\alpha)/2$. Let β denote the smallest conjugate of α^{-1} which is > 1. Then $\mathrm{M}(\alpha) = \mathrm{M}(P_\alpha^*) \geq |P_\alpha(0)|\,\beta^{\deg(\alpha)/2}$. All the roots of P_α^* are real. The same argument (Theorem 7.3) leads to

$$h(\alpha^{-1}) = h(\alpha) \geq \frac{\mathrm{Log}\,|P_\alpha(0)|}{\deg(\alpha)} + \frac{1}{2}\,\mathrm{Log}\,\theta_{31}^{-1} \geq \frac{1}{2}\,\mathrm{Log}\,\theta_{31}^{-1} = 0.04...$$

(ii) The case where α is a totally real algebraic integer $\neq 0, \neq \pm 1$ having a minimal polynomial P_α not totally positive is deduced from (i). Indeed, the polynomial $(-1)^{\deg(\alpha)} P_\alpha(x) P_\alpha(-x)$ is totally positive, of degree $2\deg(\alpha)$, and its Mahler measure is equal to $\mathrm{M}(P_\alpha)^2$. If $P_\alpha(x)P_\alpha(-x)$ has a number of roots > 1 greater than, or equal to, $\deg(\alpha)$, then β^2 denotes the smallest root of $P_\alpha(x)P_\alpha(-x)$ which is > 1. If not, β^2 denotes the smallest root of $x^{2\deg(\alpha)} P_\alpha(x^{-1}) P_\alpha(-x^{-1})$ which is > 1. Then, as above: $\mathrm{M}(\alpha)^2 \geq (\beta^2)^{\deg(\alpha)}$ with $\beta^2 > \theta_{31}^{-1}$. We deduce the minoration of $h(\alpha)$ in (1.22).
(iii) Let α be a totally real algebraic number $\neq 0, \neq \pm 1$ which is a noninteger. Let $P_\alpha(x) = c \prod_{i=1}^{\deg(\alpha)} (x - \alpha^{(i)})$ denote the minimal polynomial of α, for some integer $c \geq 2$. Using (i) and Theorem 7.3, with $P_\alpha(x)$ totally positive and reciprocal, the Mahler measure of α satisfies: $\mathrm{M}(P_\alpha) \geq c\,\beta^{\deg(\alpha)/2}$ where β is the smallest conjugate

of α which is > 1, and $\beta > \theta_{31}^{-1}$. Hence,

$$h(\alpha) \geq \frac{\operatorname{Log} c}{\deg(\alpha)} + \frac{1}{2} \operatorname{Log} \theta_{31}^{-1} \geq \frac{1}{2} \operatorname{Log} \theta_{31}^{-1}.$$

If P_α is totally positive and nonreciprocal we conclude as in (i). If P_α is not totally positive, we invoke the same arguments as in (ii). Hence (1.22) holds for all nonzero totally real algebraic numbers $\alpha \neq \pm 1$.

8 Sequences of small algebraic integers converging to 1^+ in modulus and limit equidistribution of conjugates on the unit circle. Proof of Theorem 1.7

Given a convergent sequence of algebraic numbers the limit equidistribution of the conjugates [Bu] [CLr] [DGS] [DNS] [FLr] [HNi] [Pe] [Pe2] [Pr] [Rly] often relies upon the Erdős-Turán-Amoroso-Mignotte theory [AM] [Bebu] [ET] [G] [Mt4] [VG3]. We will make use of Belotserkovski's Theorem [Bebu], recalled below as Theorem 8.1, which prefigurates Bilu's theorem on the n-dimensional torus [Bu]; the discrepancy function of equidistribution given by this theorem is well adapted to become a function of only the dynamical degree.

Theorem 8.1 (Belotserkovski). *Let $F(x) = a \prod_{i=1}^{m} (x - \alpha^{(i)}) \in \mathbb{C}[x], m \geq 1$, be a polynomial with roots $\alpha^{(k)} = r_k e^{i\varphi_k}$, $0 \leq \varphi_k \leq 2\pi$. For $0 \leq \varphi \leq \psi \leq 2\pi$, denote $N_F(\varphi, \psi) = Card\{k \mid \varphi \leq \varphi_k \leq \psi\}$. Let $0 \leq \varepsilon, \delta \leq 1/2$ and*

$$\sigma_{dis} = \max\left(m^{-1/2} \operatorname{Log}(m+1), \sqrt{-\varepsilon \operatorname{Log}(\varepsilon)}, \sqrt{-\delta \operatorname{Log}(\delta)}\right).$$

If $|r_k - 1| \leq \varepsilon$ for $1 \leq k \leq m$, and $|\operatorname{Log} a| \leq \delta m$ are satisfied, then, for some (universal, in the sense that it does not depend upon F) constant $C > 0$,

$$\left| \frac{1}{m} N_F(\varphi, \psi) - \frac{\psi - \varphi}{2\pi} \right| \leq C \sigma_{dis} \qquad \text{for all } 0 \leq \varphi \leq \psi \leq 2\pi. \tag{8.1}$$

The multiplicative group of nonzero elements of $\mathbb{C}$, resp. $\mathbb{Q}$, is denoted by $\mathbb{C}^\times$, resp. $\mathbb{Q}^\times$. The unit Dirac measure supported at $\omega \in \mathbb{C}$ is denoted by δ_ω. We denote by $\mu_{\mathbb{T}}$ the (normalized) Haar measure (unit Borel measure), invariant by rotation, that is supported on the unit circle $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$, compact subgroup of $\mathbb{C}^\times$, i.e. with $\mu_{\mathbb{T}}(\mathbb{T}) = 1$. Given $\alpha \in \overline{\mathbb{Q}}^\times$, of degree $m = \deg(\alpha)$, we define the unit Borel measure (probability)

$$\mu_\alpha = \frac{1}{\deg(\alpha)} \sum_{j=1}^{\deg(\alpha)} \delta_{\sigma(\alpha)}$$

on $\mathbb{C}^{\times}$, the sum being taken over all m embeddings $\sigma : \mathbb{Q}(\alpha) \to \mathbb{C}$. A sequence $\{\gamma_s\}$ of points of $\overline{\mathbb{Q}}^{\times}$ is said to be *strict* if any proper subgroup of $\overline{\mathbb{Q}}^{\times}$ contains γ_s for only finitely many values of s.

Theorem 6.2 in [VG6] shows that limit equidistribution of conjugates occurs on the unit circle for the sequence of Perron numbers $\{\theta_n^{-1} \mid n = 2,3,4,\ldots\}$, as $\mu_{\theta_n^{-1}} \to \mu_{\mathbb{T}}, n \to \infty$. All these Perron numbers have a Mahler measure $> \Theta$. We now give a generalization of this limit result to convergent sequences of algebraic integers of small Mahler measure, $< \Theta$, where "*convergence to* 1" has to be taken in the sense of the "house". The Theorem is Theorem 1.7.

Proof of Theorem 1.7: (i) Denote generically by $\alpha \in \mathcal{O}_{\overline{\mathbb{Q}}}$ any element of $(\alpha_q)_{q\geq 1}$. Let $m = \deg(\alpha)$ and $\beta = \lceil\overline{\alpha}\rceil \in (\theta_n^{-1}, \theta_{n-1}^{-1}), n \geq 260$. Using the inequality (6.12) between m and the dynamical degree $n = \mathrm{dyg}(\alpha) = \mathrm{dyg}(\beta)$, there exists a constant $c_1 > 0$ such that

$$\frac{\mathrm{Log}\,(m+1)}{\sqrt{m}} \leq c_1 \frac{\mathrm{Log}\, n}{\sqrt{n}}.$$

On the other hand, the minimal polynomial $P_\alpha = P_\beta$ is reciprocal and all its roots $\alpha^{(k)}$, including β and $1/\beta$ by Theorem 6.1, lie in the annulus $\{z \mid \frac{1}{\beta} \leq |z| \leq \beta\}$. As a consequence, using Theorem 5.2, there exists a constant $c_2 > 0$ such that

$$||\alpha^{(k)}| - 1| \leq \varepsilon, \quad 1 \leq k \leq m, \quad \text{with } \varepsilon = c_2 \frac{\mathrm{Log}\, n}{n}.$$

We take $\delta = 0$ in the definition of σ in Theorem 8.1 since P_α is monic. We deduce that the discrepancy function, i.e. the upper bound in the rhs of (8.1), is equal to $C\sigma_{dis} = c_3 \frac{\mathrm{Log}\, n}{\sqrt{n}}$ for some constant $c_3 > 0$. Hence,

$$\left| \frac{1}{m} N_{P_\alpha}(\varphi, \psi) - \frac{\psi - \varphi}{2\pi} \right| \leq c_3 \frac{\mathrm{Log}\, n}{\sqrt{n}} \qquad \text{for all } 0 \leq \varphi \leq \psi \leq 2\pi. \tag{8.2}$$

The discrepancy function of (8.2) tends to 0 if n tends to infinity. By Theorem 5.2 and Theorem 5.3, for $1 < \beta < \theta_{260}^{-1}$,

$$\beta \to 1^+ \Longleftrightarrow n = \mathrm{dyg}(\beta) \to \infty,$$

so that the sequence of Galois orbit measures in (1.23) converge for the weak topology as a function of the dynamical degree.

(ii) The sequence (α_q) is strict since the sequence $(\lceil\overline{\alpha_q}\rceil)$ only admits 1 as limit point: $\limsup_{q\to\infty} \lceil\overline{\alpha_q}\rceil = \lim_{q\to\infty} \lceil\overline{\alpha_q}\rceil = 1$ and the number $\mathrm{Card}\{\alpha_q \in (\theta_n^{-1}, \theta_{n-1}^{-1})\}$ between two successive Perron numbers of (θ_n^{-1}), for every $n \geq 3$, is finite. In the space of probality measures equipped with the weak topology, the reformulation of (8.2) means (1.23), equivalently (1.24).

9 Some consequences in Geometry

The reader can now transform the following Conjectures into Theorems. Some Conjectures about the problems of Lehmer in higher dimension (§ 2.2) will be reconsidered by the author later on.

Lehmer's Conjecture for Salem numbers is called "no small Salem number Conjecture", or "Salem's Conjecture" for short, in Breuillard and Deroin [BdDn].

Theorem 9.1 (Sury [Sry]). *Salem's Conjecture is true if and only if there exists a neighbourhood W of the identity in $SL(2,\mathbb{R})$ such that, for all cocompact arithmetic Fuchsian groups Γ, the intersection $\Gamma \cap W$ consists only of elements of finite order.*

This beautiful geometric reformulation (Ghate and Hironaka [GH] § 3.5, Mclachlan and Reid [MlK] pp 378-380, Margulis [Mis]) is rephrased in the equivalent statement in [BdDn]:

Salem's Conjecture holds if and only if there is a uniform positive lower bound on the length of closed geodesics in arithmetic hyperbolic 2-orbifolds.

We refer the reader to [BdDn] for the definitions of the terms used in the following Theorem, Breuillard and Deroin obtain the spectral reformulation of Salem's Conjecture, using the Cheeger-Buser inequality:

Theorem 9.2 (Breuillard - Deroin). *Salem's Conjecture holds if and only if there exists a uniform constant $c > 0$ such that*

$$\lambda_1(\widetilde{\Sigma}) \geq \frac{c}{\operatorname{area}(\widetilde{\Sigma})}$$

for all 2-covers $\widetilde{\Sigma}$ of all compact congruence arithmetic hyperbolic 2-orbifolds Σ.

In [GH] p. 304, Ghate and Hironaka mention that if Lehmer's Conjecture is true, then the following Conjecture is also true:

Conjecture 17. *(Margulis) Let G be a connected semi-simple group over $\mathbb{R}$, having $\mathrm{rank}_{\mathbb{R}}(G) \geq 2$. Then there is a neighbourhood $U \subset G(\mathbb{R})$ of the identity such that for any irreducible cocompact lattice $\Gamma \subset G(\mathbb{R})$, the intersection $\Gamma \cap U$ consists only of elements of finite order.*

Let us give a proof of Conjecture 17 from the two arguments of Margulis ([Mis], Theorem (B) p. 322): (i) first, the arithmeticity Theorem 1.16 in [Mis], p. 299, and (ii) the following statement (Margulis [Mis], p. 322):

Let $P(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_0$ be an irreducible monic polynomial with integral coefficients. Denote by $\beta_1(P), \ldots, \beta_n(P)$ the roots of P and by $m(P)$ the number of those i with $1 \leq i \leq n$ and $|\beta_i(P)| \neq 1$. Then

$$\mathrm{M}(P) = \prod_{1 \leq i \leq n} \max\{1, |\beta_i(P)|\} > d \tag{9.1}$$

where the constant $d > 1$ depends only upon $m(P)$ (and does not depend upon n). The minorant d is universal and is given by Theorem 1.2 (ex-Lehmer's Conjecture), hence the result.

But the dependency of the minorant d of $\mathrm{M}(P)$ in (9.1), expected by Margulis, with the number of roots $m(P)/2$ lying outside the closed unit disk, or equivalently inside the open unit disk (the polynomial P can be assumed of small Mahler measure $< \Theta$, hence reciprocal by Smyth's Theorem), and not with the degree n of P, is not clear in view of Theorem 5.15, Theorem 6.1 and Proposition 5.12. Indeed, the following minorant

$$m(P) \geq 2(1 + J_{\mathrm{dyg}(P)})$$

can only be deduced from the present study, this minorant being two times the cardinal of the lenticulus of roots associated with the dynamical degree of the house of the polynomial P; what can be said is that the degree $n = \deg(P)$ of P is not involved in this minorant of $m(P)$, and therefore that the constant d in (9.1) is likely to depend upon the dynamical degree $\mathrm{dyg}(P)$ (cf § 1 for its definition).

Ghate and Hironaka, in [GH] § 3.4, proved that an immediate consequence of Salem's Conjecture is the following Conjecture (cf the Corollaries 1 to 4 in [ERTz]):

Conjecture 18. *(Minimization Problem for Geodesics) There is a geodesic of minimal length amongst all closed geodesics on all arithmetic hyperbolic surfaces.*

More numerically, for short geodesics, Neumann and Reid (§ 4.4 and 4.5 in [NmRd]) proved: if Lehmer's Conjecture is true, then the following Conjecture is true.

Conjecture 19. *(Neuman-Reid) There is a universal lower bound for the lengths of geodesics in closed arithmetic hyperbolic orbifolds (the current guess is approximately 0.09174218, or twice this, 0.18348436, if the orbifold is derived from a quaternion algebra).*

In [ERTz] Emery, Ratcliffe and Tschantz show the important role played by the smallness of Salem numbers in the existence of lower bounds:

Theorem 9.3 (Emery, Ratcliffe, Tschantz). *Let $n \geq 1$ be an integer, and*

$$b_n := \min\{\mathrm{Log}\,\lambda \mid \lambda \textit{ is a Salem number with } \deg\lambda \leq n+1\}$$

$$c_n := \min\{\frac{1}{2}\mathrm{Log}\,\lambda \mid \lambda \textit{ is a Salem number with } \deg\lambda \leq n+1$$

$$\textit{which is square-rootable over } \mathbb{Q}\}.$$

Let Γ be a non-cocompact arithmetic group of isometries of $\mathbb{H}^n$ and C is a closed geodesic in $\mathbb{H}^n/\Gamma$, then

(i) if n is even, length$(C) \geq b_n$, and this lower bound is sharp for each even $n > 0$,
(ii) if n odd and $n > 1$, length$(C) \geq c_n$, and this lower bound is sharp for each odd integer $n > 1$.

In [SWs3] Silver and Williams investigate the problem of Lehmer in terms of generalized growth rates of Lefshetz numbers of iterated pseudo-Anosov surface homeomorphisms. They prove several statements equivalent to Lehmer's Conjecture. Let us mention the following (the definitions of the terms can be found in [SWs3] [Ron]):

Theorem 9.4 (Silver-Williams [SWs3]). *Lehmer's Conjecture is true if and only if there exists $c > 0$ such that, for all fibered hyperbolic knots K in a lens space $L(n,1)$, $n > 0$, the Mahler measure of the Alexander polynomial $\Delta_K(t)$ of K satisfies:*

$$1 + c < \mathrm{M}(\Delta_K(t)).$$

Appendix: Angular asymptotic sectorization of the roots $z_{j,n}, \omega_{j,n}$, of the Parry Upper functions, in lenticular sets of zeroes – notations for transition regions

The Poincaré asymptotic expansions of the roots $z_{j,n}$ of $G_n(z) = -1 + z + z^n$, lying in the first quadrant of $\mathbb{C}$, are divergent formal series of functions of the couple of **two variables** which is:

- $\big(n, \dfrac{j}{n}\big)$, in the angular sector: $\dfrac{\pi}{2} > \arg z > 2\pi \dfrac{\mathrm{Log}\, n}{n}$,
- $\big(n, \dfrac{j}{\mathrm{Log}\, n}\big)$, in the angular sector ("bump" sector): $2\pi \dfrac{\mathrm{Log}\, n}{n} > \arg z \geq 0$.

In the bump sector (cusp sector of Solomyak's fractal $\mathscr{G}$, § 4.2.2), the roots $z_{j,n}$ are dispatched into the two subsectors:

- $2\pi \frac{\sqrt{(\mathrm{Log}\, n)(\mathrm{Log}\,\mathrm{Log}\, n)}}{n} > \arg z > 0$,
- $2\pi \frac{\mathrm{Log}\, n}{n} > \arg z > 2\pi \frac{\sqrt{(\mathrm{Log}\, n)(\mathrm{Log}\,\mathrm{Log}\, n)}}{n}$.

The relative angular size of the bump sector, as $(2\pi \frac{\mathrm{Log}\, n}{n})/(\frac{\pi}{2})$, tends to zero, as soon as n is large enough. By transition region, we mean a small neighbourhood of the argument :

$$\arg z = 2\pi \frac{\mathrm{Log}\, n}{n} \quad \text{or of} \quad 2\pi \frac{\sqrt{(\mathrm{Log}\, n)(\mathrm{Log}\,\mathrm{Log}\, n)}}{n}.$$

Outside these two transition regions, a dominant asymptotic expansion of $z_{j,n}$ exists. In a transition region an asymptotic expansion contains more n-th order terms of the same order of magnitude ($n = 2, 3, 4$). These two neighbourhoods are defined

as follows. Let $\varepsilon \in (0,1)$ small enough. Two strictly increasing sequences of real numbers $(u_n), (v_n)$ are introduced, which satisfy:

$$\lfloor n/6 \rfloor \; > \; v_n \; > \; \mathrm{Log}\, n, \quad \mathrm{Log}\, n \; > \; u_n \; > \; \sqrt{(\mathrm{Log}\, n)(\mathrm{Log}\,\mathrm{Log}\, n)}, \quad \text{for } n \geq n_0 = 18,$$

such that

$$\lim_{n\to\infty} \frac{v_n}{n} \; = \; \lim_{n\to\infty} \frac{\sqrt{(\mathrm{Log}\, n)(\mathrm{Log}\,\mathrm{Log}\, n)}}{u_n} \; = \; \lim_{n\to\infty} \frac{u_n}{\mathrm{Log}\, n} \; = \; \lim_{n\to\infty} \frac{\mathrm{Log}\, n}{v_n} \; = \; 0$$

and

$$v_n - u_n = O((\mathrm{Log}\, n)^{1+\varepsilon}) \tag{9.1}$$

with the constant 1 involved in the big O. The roots $z_{j,n}$ lying in the first transition region about $2\pi(\mathrm{Log}\, n)/n$ are such that:

$$2\pi \frac{v_n}{n} \; > \; \arg z_{j,n} \; > \; 2\pi \frac{(2\mathrm{Log}\, n - v_n)}{n},$$

and the roots $z_{j,n}$ lying in the second transition region about $\frac{2\pi\sqrt{(\mathrm{Log}\, n)(\mathrm{Log}\,\mathrm{Log}\, n)}}{n}$ are such that:

$$2\pi \frac{u_n}{n} \; > \; \arg z_{j,n} \; > \; 2\pi \frac{2\sqrt{(\mathrm{Log}\, n)(\mathrm{Log}\,\mathrm{Log}\, n)} - u_n}{n}.$$

In Proposition 3.6, for simplicity's sake, these two transition regions are schematically denoted by

$$\arg z \asymp 2\pi \frac{(\mathrm{Log}\, n)}{n} \quad \text{resp.} \quad \arg z \asymp 2\pi \frac{\sqrt{(\mathrm{Log}\, n)(\mathrm{Log}\,\mathrm{Log}\, n)}}{n}.$$

By complementarity, the other sectors are schematically written:

$$2\pi \frac{\sqrt{(\mathrm{Log}\, n)(\mathrm{Log}\,\mathrm{Log}\, n)}}{n} \; > \; \arg z \; > \; 0$$

instead of

$$2\pi \frac{2\sqrt{(\mathrm{Log}\, n)(\mathrm{Log}\,\mathrm{Log}\, n)} - u_n}{n} \; > \; \arg z \; > \; 0;$$

resp.

$$2\pi \frac{\mathrm{Log}\, n}{n} \; > \; \arg z \; > \; 2\pi \frac{\sqrt{(\mathrm{Log}\, n)(\mathrm{Log}\,\mathrm{Log}\, n)}}{n}$$

instead of

$$2\pi \frac{2\mathrm{Log}\, n - v_n}{n} \; > \; \arg z \; > \; 2\pi \frac{u_n}{n};$$

resp.

$$\frac{\pi}{2} \; > \; \arg z \; > \; 2\pi \frac{\mathrm{Log}\, n}{n} \quad \text{instead of} \quad \frac{\pi}{2} \; > \; \arg z \; > \; 2\pi \frac{v_n}{n}.$$

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